Topics in Applied Math: Logic and Foundations of Mathematics

Part 5. Models of first-order arithmetic

Kazuyuki Tanaka

BIMSA

November 12, 2025

清華大学 求真书院
Qiuzhen College, Tsinghua University

Logic and Foundations

- **Part 1. Equational theory**
- **Part 2. First order theory**
- **Part 3. Basic Model theory**
- **Part 4. First order arithmetic and incompleteness theorems**
- **Part 5. Models of first-order arithmetic**
- **Part 6. Second order arithmetic and reverse mathematics**

Part 5. Schedule

- Nov. 5, (0) Nonstandard models and Overspill
- Nov. 7, (1) The omitting type theorem
- Nov.12, (2) Recursively saturated models
- Nov.14, (3) Friedman's theorem
- Nov.19, (4) Resplendency

§1. Recap

- The order type of a non-standard model of PA^- is $\mathbb{N} + \mathbb{Z} \bullet \eta$, where η is a linear ordering without a maximal element.

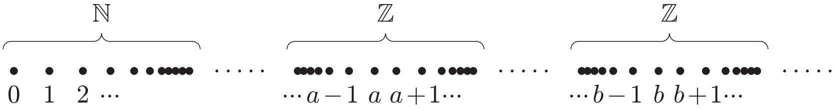

Figure: The order type of a non-standard model of arithmetic

- The order type of a non-standard model of $\mathrm{I}\Sigma_0$ is $\mathbb{N} + \mathbb{Z} \bullet \eta$, where η is a dense linear order. In particular, the order type of a countable non-standard model of $\mathrm{I}\Sigma_0$ is $\mathbb{N} + \mathbb{Z} \bullet \mathbb{Q}$.
- There is no non-standard model of PA^- with the order type $\mathbb{N} + \mathbb{Z} \bullet \mathbb{R}$.

Theorem 1.3 (Overspill principle)

Let $n > 0$ and $\mathfrak{A}$ be any non-standard model of $\mathrm{I}\Sigma_n$, and $\varphi(x)$ be any Σ_n formula. If $\mathfrak{A}_A \models \varphi(i)$ holds for infinitely many $i \in \mathbb{N}$, then there exists a non-standard element a such that $\mathfrak{A}_A \models \varphi(a)$ holds.

§2. Recap

Definition 2.1

- Let $\mathcal{L}$ be any language.
- A set $\Phi(\vec{x})$ of $\mathcal{L}$ formulas that have no free variables other than n variables $\vec{x} = (x_1, \cdots, x_n)$ is called an **n-type**, or simply a **type**.
- If n elements $\vec{a} = (a_1, \cdots, a_n)$ of $\mathcal{L}$ structure $\mathfrak{A}$ satisfies all formulas $\varphi(\vec{x})$ in $\Phi(\vec{x})$ ($\mathfrak{A}_A \models \varphi(\vec{a})$), we say that $\mathfrak{A}$ **realizes** $\Phi(\vec{x})$ by $\vec{a}$.
- If $\mathfrak{A}$ does not realize $\Phi(\vec{x})$ by any $\vec{a}$, we say that $\mathfrak{A}$ **omits** $\Phi(\vec{x})$.

Definition 2.2

- Let T be a theory in language $\mathcal{L}$.
- A type $\Phi(\vec{x})$ is called a **type of theory T** if $T \cup \Phi(\vec{c})$($\vec{c}$ are new constants) is consistent. That is, there exists a model of T that realizes $\Phi(\vec{x})$.
- Let $\mathfrak{A}$ be an $\mathcal{L}$-structure, and let C be a subset of the universe of $\mathfrak{A}$. A **type on C in a structure $\mathfrak{A}$** is a type of theory $\mathrm{Th}(\mathfrak{A}_C)$ in language $\mathcal{L}_C$. A type on $C = \varnothing$ is simply called a type.

Definition 2.4

- A type $\Phi(\vec{x})$ in $\mathcal{L}$ is called a **principal** type of theory T, if there exists a formula $\psi(\vec{x})$ in $\mathcal{L}$ such that $T \cup \{\exists \vec{x} \psi(\vec{x})\}$ is consistent, and for any $\varphi(\vec{x}) \in \Phi(\vec{x})$,

$$T \vdash \forall \vec{x}(\psi(\vec{x}) \to \varphi(\vec{x})).$$

- In this case, we say that $\psi(\vec{x})$ **generates** $\Phi(\vec{x})$ in T.
- A **non-principal** type of T is a type of T but not principal.
- A type $\Phi(\vec{x})$ on $C(\subseteq A)$ in $\mathcal{L}$-structure $\mathfrak{A}$, i.e., a type of theory $\mathrm{Th}(\mathfrak{A}_C)$ in language $\mathcal{L}_C$, is a **principal** type, if it is a principal type of theory $\mathrm{Th}(\mathfrak{A}_A)$ in language $\mathcal{L}_A$.

We now prove that there is a model of T that omits any non-principal type of T.

Theorem 2.5 (The omitting type theorem)

Let $\mathcal{L}$ be a countable language and T be a consistent theory in a language $\mathcal{L}$. Given countably many non-principal types $\Phi_i(x_1, \cdots, x_{n_i})$ of T $(i \in \mathbb{N})$, then there is a countable model of T that omits all Φ_i.

Definition 2.6

Let $\mathfrak{A}$, $\mathfrak{B}$ be two structures in a language $\mathcal{L}$ with a binary relational symbol $<$ such that $\mathfrak{B}$ is a substructure of $\mathfrak{A}$. Then, $\mathfrak{A}$ is an **end-extension** of $\mathfrak{B}$, denoted as $\mathfrak{B} \subseteq_e \mathfrak{A}$, if

$$(b \in |\mathfrak{B}| \wedge \mathfrak{A} \models a < b) \Rightarrow a \in |\mathfrak{B}|.$$

If $\mathfrak{B}$ is an elementary substructure of $\mathfrak{A}$, and $\mathfrak{A}$ is an end-extension of $\mathfrak{B}$, then $\mathfrak{A}$ is an **elementary end-extension** of $\mathfrak{B}$.

Definition 2.7

In a language $\mathcal{L}$ with a binary relation $<$, the following schema is called **collection principle**:

$$\forall x < u \exists y_1 \cdots \exists y_k \varphi(x, y_1, \cdots, y_k) \to \exists v \forall x < u \exists y_1 < v \cdots \exists y_k < v \varphi(x, y_1, \cdots, y_k).$$

where $\varphi(x, y_1, \cdots, y_k)$ is any formula in $\mathcal{L}$, and may include variables other than v.

- The collection principle holds in PA as shown in the last part.
- In set theory, when $<$ is interpreted as $\in$, it is kind of Fraenkel's axiom (also called the "replacement axiom"). Even if we interpret $<$ as $\subsetneq$ in set theory, the collection principle can be proved (in the ZF set theory).

Theorem 2.8

In a countable language $\mathcal{L}$ containing a binary relation symbol $<$, a countably infinite structure that satisfies the collection principle and the transitivity law has a proper elementary end-extension.

Corollary 2.9

A countable model of Peano arithmetic PA has a proper elementary end-extension.

- The above corollary can also be extended to non-countable models, which is called the MacDowell-Specker Theorem. For more details, see Kaye's book *Models of Peano arithmetic*.
- The proof of elementary end-extension for ZF set theory can be found in Chang and Keisler's book *Model theory*. It is also known that the results of set theory cannot be extended to non-countable cases.

Problem 2: HW # 4-2

Show that if a model $\mathfrak{A}$ of $\mathrm{I}\Sigma_0$ has a proper elementary end-extension, $\mathfrak{A}$ is a model of PA.

§3. Recursively saturated models

- Next, we want to construct countable structures that realize as many types $\Phi(\vec{x})$ as possible.
- Even if the language is countable (and so the set of formulas is countable), there can be uncountable many types $\Phi(\vec{x})$, and then it is impossible to realize all of them in countable structures.
- This brings us to the notion of "recursive saturated model", which realizes only the recursive types. Using this model, we prove "Friedman's self-embedding theorem," a groundbreaking discovery on countable non-standard models of arithmetic.
- In a countable language, the type $\Phi(\vec{x})$ is said to be recursive if the set of Gödel numbers of its formulas is recursive (computable).
- By an argument similar to Craig's Lemma in last part, the class of types is essentially the same whether they are CE, recursive, or primitive recursive.

Definition 3.1

Let $\mathcal{L}$ be a countable language. An $\mathcal{L}$-structure $\mathfrak{A}$ is **recursively saturated** if any recursive 1-type over a finite set $\{a_1, \cdots, a_n\} \subseteq A$ is realized in $\mathfrak{A}$, that is, any recursive type $\Phi(x_0, x_1, \cdots, x_n) = \{\varphi_i(x_0, x_1, \cdots, x_n) \mid i \in \mathbb{N}\}$ and for any $a_1, \cdots, a_n \in A$,

$$\forall j \exists a \in A \forall i < j\, \mathfrak{A}_A \models \varphi_i(a, a_1, \cdots, a_n) \Rightarrow \exists a \in A \forall i\, \mathfrak{A}_A \models \varphi_i(a, a_1, \cdots, a_n).$$

Problem 3

Show that any finite structure is recursively saturated.

- The standard structure of arithmetic $\mathfrak{N}$ is clearly not recursively saturated. However, by the next lemma, there exists a recursively saturated countable non-standard model that is elementary equivalent to $\mathfrak{N}$.

Lemma 3.2

A countable structure in a countable language has a countable elementary extension which is recursively saturated.

Proof.

- Let $\mathfrak{A}$ be a countable structure in a countable language. For each recursive type $\Phi = \{\varphi_i(x_0, x_1, \cdots, x_n) \mid i \in \mathbb{N}\}$ and for each $a_1, \cdots, a_n \in A$, we add a new constant $c_{\Phi,a_1,\cdots,a_n}$ to the language, and let

$$T_1 = \mathrm{Th}(\mathfrak{A}_A) \cup \{\exists x \forall i < j \varphi_i(x, a_1, \cdots, a_n) \to \forall i < j \varphi_i(c_{\Phi,a_1,\cdots,a_n}, a_1, \cdots, a_n) : j \in \mathbb{N} \text{ and } c_{\Phi,a_1,\cdots,a_n} \text{is a new constant for any } \Phi, a_1, \cdots, a_n \in A\}.$$

- By the compactness theorem and the downward Löwenheim–Skolem Theorem, T_1 has a countable model $\mathfrak{A}_1$.
- Then $\mathfrak{A} \prec \mathfrak{A}_1$ and $\mathfrak{A}_1$ realizes all recursive 1-types over any finite subset of A.
- Next, we construct a countable model $\mathfrak{A}_2 \succ \mathfrak{A}_1$ that realizes all recursive 1-types over a finite subset of A_1.

- Similarly, we create $\mathfrak{A}_2 \prec \mathfrak{A}_3 \prec \mathfrak{A}_4 \prec \cdots$, and denote $\mathfrak{A}_\infty = \bigcup_k \mathfrak{A}_k$.
- By the elementary chain theorem in part 3, $\mathfrak{A}_\infty$ is an elementary extension of $\mathfrak{A}$ and is also countable.

Elementary chain theorem

Let $\mathfrak{A}_0 \prec \mathfrak{A}_1 \prec \cdots$ be an elementary chain. Let $\mathfrak{A}$ be the union of the elementary chain. Then for each i, $\mathfrak{A}_i \prec \mathfrak{A}$.

- To see that $\mathfrak{A}_\infty$ is recursively saturated, we arbitrarily select a finite number of elements from $\mathfrak{A}_\infty$ and consider a recursive type over them.
- It is a type over A_k for a sufficiently large k, and is realized by $\mathfrak{A}_{k+1}$, and also by its elementary extension $\mathfrak{A}_\infty$. □

Now we will consider models of arithmetic $\mathrm{I}\Sigma_n$. Although these models are not necessarily recursively saturated, they have a certain kind of saturation for a restricted class of formulas, and their properties are deeply related with the relations.

Lemma, revisiting Lem 4.4.13

In a consistent Σ_1-complete theory T, there exists no formula $\psi(x)$ such that for any sentence σ, $T \vdash \sigma \leftrightarrow \psi(\overline{\ulcorner \sigma \urcorner})$.

First, let us rephrase the above lemma as follows.

Lemma 3.3 (Tarski's "undefinability of truth")

Let T be a consistent extension of $\mathrm{Q}_<$. There is no formula $\mathrm{Sat}(x, y)$ such that: for any $\mathcal{L}_{\mathrm{OR}}$ formula $\varphi(v_1, \cdots, v_k)$ (with only free variables $v_1, \cdots, v_k$),

$$T \vdash \forall s(\mathrm{Sat}(\overline{\ulcorner \varphi \urcorner}, s) \leftrightarrow \varphi(s_1, \cdots, s_k)),$$

where s is the code of a sequence $(s_1, \cdots, s_k)$.

Since the revisited lemma states that $\mathrm{Sat}(x, \varnothing)$ does not exist, the above lemma can be derived immediately. But if we restrict $\varphi(v_1, \cdots, v_k)$ to Σ_n, a kind of $\mathrm{Sat}(x, y)$ exists.

Lemma 3.4

For each $n \in \mathbb{N}$, there exist formulas $\mathrm{Sat}_{\Sigma_n}(x,y)$ and $\mathrm{Sat}_{\Pi_n}(x,y)$ in language $\mathcal{L}_{\mathrm{OR}}$ such that for any Σ_n formula $\varphi(v_1,\cdots,v_k)$ and Π_n formula $\psi(v_1,\cdots,v_k)$ (neither includes free variables other than $v_1,\cdots,v_k$),

$$\mathrm{I}\Sigma_1 \vdash \forall s(\mathrm{Sat}_{\Sigma_n}(\overline{\ulcorner\varphi\urcorner}, s) \leftrightarrow \varphi(s_1,\cdots,s_k)),$$
$$\mathrm{I}\Sigma_1 \vdash \forall s(\mathrm{Sat}_{\Pi_n}(\overline{\ulcorner\psi\urcorner}, s) \leftrightarrow \psi(s_1,\cdots,s_k)),$$

where s is the code of $(s_1,\cdots,s_k)$. When $n>0$, $\mathrm{Sat}_{\Sigma_n} \in \Sigma_n$ and $\mathrm{Sat}_{\Pi_n} \in \Pi_n$.

Note that considering $\mathrm{Bew}_T(x)$ in the proof of the second incompleteness theorem in part 4 of this course, it can be shown that for Σ_1 formula $\varphi(v)$,

$$\varphi(v) \to \mathrm{Bew}_T(\overline{\ulcorner\dot{v}\urcorner}),$$

but the inverse $\leftarrow$ does not hold. In particular, if φ is $0=1$, $\mathrm{Bew}_T(\overline{\ulcorner 0=1\urcorner}) \to 0=1$ is nothing but Con(T).

Proof.

- To start with, consider the case where $n = 0$. Roughly speaking, the truth of a Σ_0 sentence is defined primitive-recursively, and so by the following theorem, Sat_{Σ_0} can be expressed by either Σ_1 or Π_1 in $\mathrm{I}\Sigma_1$.

Definability theorem of primitive recursive functions, revisiting Th.4.3.6

In IΣ_1, the graph of a primitive recursive function $f(x_1, \cdots, x_l, y) = z$ can be represented by a Δ_1 formula $\varphi(x_1, \ldots, x_l, y, z)$, and the following is provable

$$\forall x_1 \cdots \forall x_l \forall y \exists ! z \varphi(x_1, \ldots, x_l, y, z).$$

- We will check more details in its construction.
- First we consider the atomic formula in the form of $u = t(v_1, \cdots, v_k)$, where t is a term that does not include free variables other than $v_1, \cdots, v_k$, and u is a variable.

- List all the subterms of t appropriately as $t_0, t_1, \cdots, t_{l_t}$. We may assume that the relation between $\ulcorner t \urcorner$ and the sequence $(\ulcorner t_0 \urcorner, \cdots, \ulcorner t_{l_t} \urcorner)$ is primitive recursive. Then, we define Sat_{Σ_0} as follows:

$$\begin{aligned}
&\mathrm{Sat}_{\Sigma_0}(\ulcorner u = t(v_1, \cdots, v_k) \urcorner, (y, x_1, \cdots, x_k)) \\
&\quad \leftrightarrow \exists z (z = (z_0, z_1, \cdots, z_{l_t}) \land \forall i, i', i'' \leq l_t \\
&\qquad\quad ((\ulcorner t_i \urcorner = \ulcorner 0 \urcorner \to z_i = 0) \land (\ulcorner t_i \urcorner = \ulcorner 1 \urcorner \to z_i = 1) \\
&\qquad \land (\ulcorner t_i \urcorner = \ulcorner v_{i'} \urcorner \to z_i = x_{i'}) \\
&\qquad \land (\ulcorner t_i \urcorner = \ulcorner t_{i'} + t_{i''} \urcorner \to z_i = z_{i'} + z_{i''}) \\
&\qquad \land (\ulcorner t_i \urcorner = \ulcorner t_{i'} \bullet t_{i''} \urcorner \to z_i = z_{i'} \bullet z_{i''}) \\
&\qquad \land (\ulcorner t_i \urcorner = \ulcorner t \urcorner \to z_i = y)))
\end{aligned}$$

- The above Σ_1 formula can be also expressed as a Π_1 formula in the form of $(\forall z (z = (z_0, z_1, \cdots, z_{l_t}) \to \cdots))$. So, we may take Sat_{Σ_0} as a Δ_1 formula.
- It is obvious that for a Σ_0 formula $u = t(v_1, \cdots, v_k)$, $\mathrm{Sat}_{\Sigma_0}(\ulcorner u = t(v_1, \cdots, v_k) \urcorner, (y, x_1, \ldots, x_k))$ and $y = t(x_1, \cdots, x_k))$ are equivalent over $\mathrm{I}\Sigma_1$.

- For a general Σ_0 formula, we can decomposed it into subformulas so that each part satisfies the conditions (Tarski's truth clauses). For more details, refer to Kaye's book *Models of Peano arithmetic*. Note that since $\Sigma_0 = \Pi_0$, Sat_{Π_0} is also equivalent to Sat_{Σ_0} which is Δ_1 .
- Next, by induction on the meta-variable n, we construct $\mathrm{Sat}_{\Sigma_{n+1}}$ assuming Sat_{Σ_n} is already obtained. For a Σ_{n+1} formula $\exists u_1 \cdots \exists u_j \varphi(u_1, \cdots, u_j, v_1, \cdots, v_k)$ (where, $\varphi \in \Pi_n$), $\mathrm{Sat}_{\Sigma_{n+1}}$ is defined as follows.

$$\begin{aligned} &\mathrm{Sat}_{\Sigma_{n+1}}(\ulcorner \exists u_1 \cdots \exists u_j \varphi(u_1, \cdots, u_j, v_1, \cdots, v_k) \urcorner, (s_1, \cdots, s_k)) \\ &\quad \leftrightarrow \exists y \mathrm{Sat}_{\Pi_n}(\ulcorner \varphi(u_1, \cdots, u_j, v_1, \cdots, v_k) \urcorner, (y_1, \cdots, y_j, s_1, \cdots, s_k)). \end{aligned}$$

- Then the following is provable in $\mathrm{I}\Sigma_1$.

$$\begin{aligned} &\mathrm{Sat}_{\Sigma_{n+1}}(\overline{\ulcorner \exists u_1 \cdots \exists u_j \varphi(u_1, \cdots, u_j, v_1, \cdots, v_k) \urcorner}, (s_1, \cdots, s_k)) \\ &\leftrightarrow \exists y \mathrm{Sat}_{\Pi_n}(\overline{\ulcorner \varphi(u_1, \cdots, u_j, v_1, \cdots, v_k) \urcorner}, (y_1, \cdots, y_j, s_1, \cdots, s_k)) \\ &\leftrightarrow \exists y \varphi(y_1, \cdots, y_j, s_1, \cdots, s_k) \\ &\leftrightarrow \exists u_1 \cdots \exists u_j \varphi(u_1, \cdots, u_j, s_1, \cdots, s_k). \end{aligned}$$

- Finally, $\mathrm{Sat}_{\Pi_{n+1}}$ can be defined in the same way. □

There is a close relation between the existence of a satisfaction relation and the saturation of a model.

Lemma 3.5

For each $n > 0$, a non-standard model $\mathfrak{A}$ of $\mathrm{I}\Sigma_n$ realizes any finitely satisfiable recursive 1-type on a finite subset of A consisting of only Σ_n formulas. Then $\mathfrak{A}$ is called **Σ_n-recursively saturated**.

Proof. Let $\Phi(x_0, x_1, \cdots, x_k)$ be a recursive type consisting only of Σ_n formulas. By Craig's lemma, assume Φ is primitively recursive.

Craig's lemma, revisited

For a CE theory T, there exists a primitive recursive theory T' that proves the same theorems.

- By the definability theorem of primitive recursive functions, the Gödel number of formulas in $\Phi(x_0, x_1, \cdots, x_k)$ can be expressed by a Σ_1 formula $\varphi(x)$ and a Π_1 formula $\varphi'(x)$, whose equivalence can be proved in $\mathrm{I}\Sigma_1$.

- The finite satisfiability of $\Phi(x_0, a_1, \cdots, a_k)$ is expressed as: for each natural number j,

$$\exists x \forall i < \bar{j}(\varphi'(i) \to \mathrm{Sat}_{\Sigma_n}(i, (x, a_1, \cdots, a_k))),$$

  which is proved to be Σ_n in $\mathrm{B}\Sigma_n (\subseteq \mathrm{I}\Sigma_n)$.
- Let $\mathfrak{A}$ be a non-standard model of $\mathrm{I}\Sigma_n$. Since the above formula holds for each $j \in \mathbb{N}$, by the overspill principle, it also holds for some infinite element j'. Let $x = a$ that satisfies the above formulas for this j'.
- For any natural number i, we have $\varphi'(\bar{i}) \to \mathrm{Sat}_{\Sigma_n}(\bar{i}, (a, a_1, \cdots, a_k))$.
- Now, if i is the Gödel number of a formula in $\Phi(x_0, x_1, \cdots, x_k)$, the Σ_1 formula $\varphi(\bar{i})$ holds. So $\mathfrak{A}_A \models \mathrm{Sat}_{\Sigma_n}(\bar{i}, (a, a_1, \cdots, a_k))$. That is, $\Phi(a, a_1, \cdots, a_k)$ holds.
- Therefore, a finitely satisfiable recursive 1-type of Σ_n formulas is realized in $\mathfrak{A}$. $\square$

By the above lemma, any non-standard model of PA is Σ_n-recursively saturated for each $n > 0$, but in the next problem, we show there is a non-standard model of PA which is not recursively saturated.

If the satisfaction relation $\mathrm{Sat}(x, y)$ were defined in PA, any non-standard model of PA would be recursively saturated in the same way as in the above lemmma. So, this is another proof that the satisfaction relation is not definable in PA.

Problem 4: HW # 4-3

Let $\mathfrak{A}$ be a non-standard model of PA, and $a \in A$ be an arbitrary non-standard element. Then, in $\mathfrak{A}$, let $\mathrm{K}(\mathfrak{A}\,; a)$ denote the set of all element $b \in A$ that can be defined by the formula $\varphi(x, a)$ (does not include parameters other than a). That is, $\mathrm{K}(\mathfrak{A}\,; a)$ denote the set of b's such that $\mathfrak{A}_{\{a,b\}} \models \forall x(x = b \leftrightarrow \varphi(x, a))$. Then prove the following.
(1) By restricting functions and relations of $\mathfrak{A}$ to that of $\mathrm{K}(\mathfrak{A}\,; a)$, $\mathrm{K}(\mathfrak{A}\,; a)$ can be seen as a substructure of $\mathfrak{A}$. $\mathrm{K}(\mathfrak{A}\,; a)$ is an elementary substructure of $\mathfrak{A}$.
(2) $\Phi(x, a) = \{\exists v \varphi(v, a) \to \exists v < x \ \varphi(v, a) \ : \ \varphi(v, u)$ contains no free variables or parameters other than $u, v\}$ is recursive and finitely satisfiable, but it cannot be realized by $\mathrm{K}(\mathfrak{A}\,; a)$.

Problem 5

Let $\mathfrak{A} = (A, +, \bullet, 0, 1, <)$ be a non-standard model of $\mathrm{I}\Sigma_1$. Show that $\mathfrak{A}' = (A, +, 0, 1, <)$ is recursively saturated. You may use Presburger's theorem that any formula in the language of $\mathfrak{A}'$ can be transformed into an equivalent quantifier-free formula. The inverse of this theorem will be discussed in Example 5 in the next section.)

In the above lemma, we will extend a recursive type to a little more general class. To this end, we introduce the following concept.

Definition 3.6

Let $\mathfrak{A}$ be a model of $\mathrm{I}\Sigma_1$, and $a \in A$. The set

$$\{n \in \mathbb{N} : \mathfrak{A} \models \overline{p(n)} | a\}$$

is called the set **coded by a** in $\mathfrak{A}$, where $p(n)$ is a primitive recursive function representing the $n+1$-th prime number, and $u|v \equiv \exists w \leq v(u \bullet w = v)$. The collection of all the sets encoded by an element in $\mathfrak{A}$ is called the **standard system** of $\mathfrak{A}$, denoted as $\mathrm{SSy}(\mathfrak{A})$.

Lemma 3.7 (D. Scott)

Let $\mathfrak{A}$ be a non-standard model of $\mathrm{I}\Sigma_1$. Given two disjoint Σ_1 sets, there exists a set in $\mathrm{SSy}(\mathfrak{A})$ which separates them. In particular, any recursive set belongs to $\mathrm{SSy}(\mathfrak{A})$.

Proof.

- Let $\exists y \theta_i(x, y)$ (θ_i is a Σ_0 formula, $i = 0, 1$) represent two disjoint Σ_1 sets.
- Let $\mathfrak{A}$ be a non-standard model of $\mathrm{I}\Sigma_1$. Then consider the following Σ_1 formula:

$$\exists v \forall x, y < \overline{j}((\theta_0(x, y) \to p(x)|v) \land (\theta_1(x, y) \to p(x) \not| v)).$$

  This holds for any standard natural number j in $\mathfrak{A}$. Then by the overspill principle, it also holds for a non-standard element $j = b$.

- Let c be such that $v = c$ satisfies the above formula with $j = b$. Then, the set coded by c separates the two initially given Σ_1 sets as follows.

$$\mathfrak{N} \models \exists y\ \theta_0(\overline{n}, y) \Rightarrow \mathfrak{A}_{\{b\}} \models \exists y < b\ \theta_0(\overline{n}, y) \Rightarrow \mathfrak{A}_{\{c\}} \models \overline{p(n)}|c,$$

$$\mathfrak{N} \models \exists y\ \theta_1(\overline{n}, y) \Rightarrow \mathfrak{A}_{\{b\}} \models \exists y < b\ \theta_1(\overline{n}, y) \Rightarrow \mathfrak{A}_{\{c\}} \models \overline{p(n)} \not| c. \quad \square$$

Note that in general, a set that separates two Σ_1 sets cannot be obtained recursively. That is, $\mathrm{SSy}(\mathfrak{A})$ is properly larger than the class of recursive sets.

Lemma 3.8

Let $n > 0$ and $\mathfrak{A}$ be a non-standard model of $\mathrm{I}\Sigma_n$. If a type $\Phi(\vec{x})$ of Σ_n formulas over a finite subset of A is coded in $\mathfrak{A}$, then $\mathfrak{A}$ realizes $\Phi(\vec{x})$.

The proof is exactly the same as that of lemma in Page 17. The converse holds as follows.

Lemma 3.9

Let $n > 0$ and $\mathfrak{A}$ be a non-standard model of $\mathrm{I}\Sigma_n$. Fix $\vec{a} \in A^{<\omega}$ arbitrarily. Then the following k types can be coded.

$$\Phi(\vec{x}) = \{\varphi(\vec{x}) : \varphi(\vec{x}) \in \Sigma_n \wedge \mathfrak{A} \models \varphi(\vec{a})\},$$
$$\Psi(\vec{x}) = \{\psi(\vec{x}) : \psi(\vec{x}) \in \Pi_n \wedge \mathfrak{A} \models \psi(\vec{a})\}$$

Proof. In $\mathrm{I}\Sigma_1$, $\mathrm{Sat}_{\Sigma_n}(x, y)$ and $\mathrm{Sat}_{\Pi_n}(x, y)$ can be defined. So, there exist Σ_n formula $\varphi_1(k, \vec{a})$ and Π_n formula $\psi_1(k, \vec{a})$ s.t. $\varphi \in \Phi \leftrightarrow \varphi_1(\ulcorner\varphi\urcorner, \vec{a})$ and $\psi \in \Psi \leftrightarrow \psi_1(\ulcorner\psi\urcorner, \vec{a})$ hold. Then, letting c be a non-standard element of $\mathfrak{A}$, by Σ_n induction, we can define a code $\Pi_{b \in U} p(b)$ for $U = \{b < c : \varphi_1(b, \vec{a})\}$ and a code $\Pi_{b \in V} p(b)$ for $V = \{b < c : \psi_1(b, \vec{a})\}$. It is clear that these code $\Phi(\vec{x})$ and $\Psi(\vec{x})$, respectively. □

With the above preparations, we will prove Friedman's self-embedding theorem. The following lemma is a key point, and also used in several variations of the theorem.

Lemma 3.10

Assuming $n > 0$, let $\mathfrak{A}$, $\mathfrak{B}$ be countable non-standard models of $\mathrm{I}\Sigma_n$. Take $a_0 \in A$ and $b_0, c \in B$ arbitrarily. Then the following two conditions are equivalent.

(1) There exists $\mathfrak{B}' \subseteq_e \mathfrak{B}$ such that $c \notin B'$. There is an isomorphism h between $\mathfrak{A}$ and $\mathfrak{B}'$ such that $h(a_0) = b_0$. For any Π_{n-1} formula $\varphi(\vec{x})$ and any $\vec{b} \in B'^{<\omega}$,

$$\mathfrak{B}'_{\{\vec{b}\}} \models \varphi(\vec{b}) \Leftrightarrow \mathfrak{B}_{\{\vec{b}\}} \models \varphi(\vec{b}).$$

(2) $\mathrm{SSy}(\mathfrak{A}) = \mathrm{SSy}(\mathfrak{B})$, and for any Π_{n-1} formula $\varphi(\vec{v}, u)$,

$$\mathfrak{A}_A \models \exists \vec{v} \varphi(\vec{v}, a_0) \Rightarrow \mathfrak{B}_B \models \exists \vec{v} < c\, \varphi(\vec{v}, b_0),$$

where $\vec{v} = (v_1, \ldots, v_k)$ and $\exists \vec{v} < c$ means $\exists v_1 < c \cdots \exists v_k < c$.

Proof. Assume (1) and we show the first half of (2).

- By $\mathfrak{A} \cong \mathfrak{B}'$, $\mathrm{SSy}(\mathfrak{A}) = \mathrm{SSy}(\mathfrak{B}')$ is obvious.
- Since $\mathfrak{B}' \subseteq_e \mathfrak{B}$, it is also clear that $\mathrm{SSy}(\mathfrak{B}') \subseteq \mathrm{SSy}(\mathfrak{B})$.
- Assume that $R \in \mathrm{SSy}(\mathfrak{B})$ and R is coded by r in $\mathfrak{B}$. We will show that R is also coded in $\mathfrak{B}'$.
- Take any non-standard element l of B'. Since $\mathfrak{B}'$ is also a model of $\mathrm{I}\Sigma_n$ $(n > 0)$, the $l+1$-th prime $p(l)$ belongs to B', and $p(l)! \in B'$.
- Therefore, letting m be the greatest common divisor of r and $p(l)!$ in $\mathfrak{B}$, we have $m \in B'$ since $\mathfrak{B}'$ is an initial segment of $\mathfrak{B}$. Then, it is clear that m also encodes R.
- From the above, we obtain $\mathrm{SSy}(\mathfrak{A}) = \mathrm{SSy}(\mathfrak{B})$.

Next we show the second half of (2).

- Let $\varphi(\vec{v}, u)$ be a Π_{n-1} formula, and $\mathfrak{A}_A \models \exists \vec{v} \varphi(\vec{v}, a_0)$.
- By the isomorphism between $\mathfrak{A}$ and $\mathfrak{B}'$, $\mathfrak{B}'_{B'} \models \exists \vec{v} \varphi(\vec{v}, b_0)$.
- Then, since there exists $\vec{d} \in B'$ such that $\mathfrak{B}'_{B'} \models \varphi(\vec{d}, b_0)$, from the assumption (1), $\mathfrak{B}_B \models \varphi(\vec{d}, b_0)$. Therefore, $\mathfrak{B}_B \models \exists \vec{v} < c\, \varphi(\vec{v}, b_0)$.

Next, assuming (2), we show (1).

- This is an application of the so-called **back-and-forth argument**. We alternately produce a list $a_0, a_1, \cdots$ of the elements of A and a list $b_0, b_1, \cdots$ of the elements of B', and an isomorphism h between $\mathfrak{A}$ and $\mathfrak{B}'$ defiend by $h(a_i) = b_i$.
- Now, suppose $a_0, a_1, \cdots, a_{2k}$ and $b_0, b_1, \cdots, b_{2k}$ have been chosen, and for any Π_{n-1} formula $\varphi(\vec{v}, \vec{u})$,

$$\mathfrak{A}_A \models \exists \vec{v} \varphi(\vec{v}, a_0, \cdots, a_{2k}) \Rightarrow \mathfrak{B}_B \models \exists \vec{v} < c\, \varphi(\vec{v}, b_0, \cdots, b_{2k}) \qquad (\sharp)$$

  holds.
- We next choose a_{2k+1}, a_{2k+2} and b_{2k+1}, b_{2k+2} such that this condition is preserved. We will explain later that (1) can be obtained by this.
- Since A is countable, each member can be assigned by a natural number uniquely. Then choose one with the smallest number among the elements that do not appear in $a_0, a_1, \cdots, a_{2k}$ and denote it as a_{2k+1}. This process guarantees that $\{a_i : i \in \mathbb{N}\}$ lists all the members of A.

- Now we will search for b_{2k+1} such that ($\sharp$) holds.
- Let $\Phi(\vec{x})$ be the set of Σ_n formulas $\exists \vec{v} \varphi(\vec{v}, x_0, \cdots, x_{2k+1})$ ($\varphi \in \Pi_{n-1}$) which holds for $a_0, \cdots, a_{2k}, a_{2k+1}$ in $\mathfrak{A}$. By the second lemma in page 27, $\Phi(\vec{x})$ is coded in $\mathfrak{A}$. Since $\mathrm{SSy}(\mathfrak{A}) = \mathrm{SSy}(\mathfrak{B})$, so it is also coded in $\mathfrak{B}$.
- Furthermore, we let

$$\begin{aligned}&\Phi'(x_0, \cdots, x_{2k+1}, x_{2k+2})\\ &= \{\exists \vec{v} < x_{2k+2}\, \varphi(\vec{v}, x_0, \cdots, x_{2k+1}) : \exists \vec{v} \varphi(\vec{v}, x_0, \cdots, x_{2k+1}) \in \Phi\}.\end{aligned}$$

  Since there is a primitive recursive transformation between Φ and Φ', Φ' is also coded in $\mathfrak{B}$.
- Then, if $\Phi'(b_0, \cdots, b_{2k}, x, c)$ is shown to be finitely satisfiable in $\mathfrak{B}$, then by the lemma in page 17, we can find an element $x = b$ that realizes $\Phi'(b_0, \cdots, b_{2k}, x, c)$, and letting b_{2k+1} be such a b, ($\sharp$) holds.
- Now, let $\exists \vec{v} < c\, \varphi_i(\vec{v}, b_0, \cdots, b_{2k}, x)$ $(i \leq j)$ be any finite set of formulas from $\Phi'(b_0, \cdots, b_{2k}, x, c)$.

- From the definition of Φ', for each $i \le j$, $\exists \vec{v} \varphi_i(\vec{v}, a_0, \cdots, a_{2k}, a_{2k+1})$ holds in $\mathfrak{A}$, so

$$\mathfrak{A}_A \models \exists \vec{v}_0 \cdots \exists \vec{v}_j \exists x \bigwedge_{i \le j} \varphi_i(\vec{v}_i, a_0, \cdots, a_{2k}, x).$$

- On the other hand, using ($\sharp$),

$$\mathfrak{B}_B \models \exists \vec{v}_0 < c \cdots \exists \vec{v}_j < c \exists x < c \bigwedge_{i \le j} \varphi_i(\vec{v}_i, b_0, \cdots, b_{2k}, x).$$

- Therefore, by simple transformation,

$$\mathfrak{B}_B \models \exists x \bigwedge_{i \le j} \exists \vec{v} < c \, \varphi_i(\vec{v}, b_0, \cdots, b_{2k}, x).$$

- In other words, $\Phi'(b_0, \cdots, b_{2k}, x, c)$ is finitely satisfiable, and b_{2k+1} is obtained.

- Next, we first select b_{2k+2} and we search for a corresponding a_{2k+2}. If $\{b_0, \cdots, b_{2k}, b_{2k+1}\}$ is an initial segment of $\mathfrak{B}$, then $b_{2k+2} = b_{2k+1}$, $a_{2k+2} = a_{2k+1}$, and ($\sharp$) holds.
- Otherwise, there exists a $b < \max\{b_0, \cdots, b_{2k}, b_{2k+1}\}$ such that b does not appear in $b_0, \cdots, b_{2k}, b_{2k+1}$. Then among such, let b_{2k+2} be one with the minimal number assigned in advance to the members of B. This finally produces $\{b_i : i \in \mathbb{N}\}$ as an initial segment of $\mathfrak{B}$.
- Then we will find a_{2k+2} corresponding to b_{2k+2}.
- Let $\Psi(\vec{x})$ be the set of Σ_n formulas $\forall \vec{v} < x_{2k+3}\, \psi(\vec{v}, x_0, \cdots, x_{2k+2})$ holds for $b_0, \cdots, b_{2k+1}, b_{2k+2}, c$ in $\mathfrak{B}$. This can be coded in $\mathfrak{B}$.
- Therefore, if we define

$$\begin{aligned}&\Psi'(x_0, \cdots, x_{2k+1}, x_{2k+2})\\&= \{\forall \vec{v} \psi(\vec{v}, x_0, \cdots, x_{2k+2}) : \forall \vec{v} < x_{2k+3}\, \psi(\vec{v}, x_0, \cdots, x_{2k+2}) \in \Psi\}\end{aligned}$$

  then Ψ' is coded in $\mathfrak{A}$ by the same argument as above.
- All that remains is to show $\Psi'(a_0, \cdots, a_{2k+1}, x)$ is finitely satisfiable in $\mathfrak{A}$. So, let $\forall \vec{v} \psi_i(\vec{v}, a_0, \cdots, a_{2k+1}, x)$ $(i \le j)$ be a finite subset of $\Psi'(a_0, \cdots, a_{2k+1}, x)$.

- We will show that these formulas are realized by $x = a$ such that $a < \max\{a_0, \cdots, a_{2k}, a_{2k+1}\}$.
- By way of contradiction, assume

$$\mathfrak{A}_A \models \forall x < \max\{a_0, \cdots, a_{2k}, a_{2k+1}\} \exists \vec{v} \bigvee_{i \leq j} \neg\psi_i(\vec{v}, a_0, \cdots, a_{2k+1}, x).$$

- By the Σ_n collection principle that follows from Σ_n induction,

$$\mathfrak{A}_A \models \exists y \forall x < \max\{a_0, \cdots, a_{2k}, a_{2k+1}\} \exists \vec{v} < y \bigvee_{i \leq j} \neg\psi_i(\vec{v}, a_0, \cdots, a_{2k+1}, x).$$

- On the other hand, using (♯),

$$\mathfrak{B}_B \models \exists y < c \forall x < \max\{b_0, \cdots, b_{2k}, b_{2k+1}\} \exists \vec{v} < y \bigvee_{i \leq j} \neg\psi_i(\vec{v}, b_0, \cdots, b_{2k+1}, x).$$

- Therefore, by simple transformation,

$$\mathfrak{B}_B \models \forall x < \max\{b_0, \cdots, b_{2k}, b_{2k+1}\} \exists \vec{v} < c \bigvee_{i \leq j} \neg\psi_i(\vec{v}, b_0, \cdots, b_{2k+1}, x)$$

  This is contradicts with the assumption that $b_0, \cdots, b_{2k+1}, b_{2k+2}, c$ realize $\Psi(\vec{x})$.
- Thus, $\Psi'(a_0, \cdots, a_{2k+1}, x)$ is finitely satisfiable, and so the desired a_{2k+2} exists.

- Suppose that we have completed the construction of a list $a_0, a_1, \cdots$, and a list $b_0, b_1, \cdots$. As described above, $A = \{a_i : i \in \mathbb{N}\}$ and $B' = \{b_i : i \in \mathbb{N}\}$ is an initial segment of $\mathfrak{B}$. It is also obvious that $c \notin B'$.
- Next, we define a function h between $\mathfrak{A}$ and $\mathfrak{B}'$ by $h(a_i) = b_i$. Then, h is an isomorphism, since by ($\sharp$), for an atomic formula $\varphi(x_0, \cdots, x_k)$,

$$\mathfrak{A}_A \models \varphi(a_0, \cdots, a_k) \Rightarrow \mathfrak{B}_B \models \varphi(b_0, \cdots, b_k),$$

  which implies h preserves operations and $<$.
- Moreover, by ($\sharp$), we can show that for any Π_{n-1} formula $\varphi(x_0, \cdots, x_k)$,

$$\mathfrak{A}_A \models \varphi(a_0, \cdots, a_k) \Leftrightarrow \mathfrak{B}_B \models \varphi(b_0, \cdots, b_k).$$

  $\Rightarrow$ is clear. For $\Leftarrow$, let $\mathfrak{A}_A \not\models \varphi(a_0, \cdots, a_k)$. Then $\mathfrak{A}_A \models \neg\varphi(a_0, \cdots, a_k)$, and $\neg\varphi(a_0, \cdots, a_k)$ is Σ_{n-1}, so by ($\sharp$), $\mathfrak{B}_B \models \neg\varphi(b_0, \cdots, b_k)$, and $\mathfrak{B}_B \not\models \varphi(b_0, \cdots, b_k)$.
- On the other hand, since h is isomorphic, for any formula $\varphi(x_0, \cdots, x_k)$,

$$\mathfrak{A}_A \models \varphi(a_0, \cdots, a_k) \Leftrightarrow \mathfrak{B}'_{B'} \models \varphi(b_0, \cdots, b_k).$$

  So for any Π_{n-1} formula $\varphi(x_0, \cdots, x_k)$,

$$\mathfrak{B}'_{B'} \models \varphi(b_0, \cdots, b_k) \Leftrightarrow \mathfrak{B}_{B'} \models \varphi(b_0, \cdots, b_k),$$

  and thus (1) is obtained. □

Theorem 3.11 (Friedman's self-embedding theorem)

Let $n > 0$, $\mathfrak{A}$ be a countable non-standard model of $\mathrm{I}\Sigma_n$, and take $a \in A$ arbitrarily. Then there exists an initial segment $\mathfrak{A}'$ of $\mathfrak{A}$ such that $a \in A' \subsetneq A$ and $\mathfrak{A}' \cong \mathfrak{A}$, and for any Π_{n-1} formula $\varphi(\vec{x})$ and any $\vec{a'} \in A'^{<\omega}$,

$$\mathfrak{A}'_{A'} \models \varphi(\vec{a'}) \Leftrightarrow \mathfrak{A}_{A'} \models \varphi(\vec{a'}).$$

Proof.

- In last lemma, we consider the case $\mathfrak{A} = \mathfrak{B}$. In order to satisfy the condition (2) of the last lemma, for any Π_{n-1} formula $\varphi(\vec{v}, u)$, it is sufficient to find c such that

$$\mathfrak{A}_{\{a\}} \models \exists \vec{v} \varphi(\vec{v}, a) \Rightarrow \mathfrak{A}_{\{a,c\}} \models \exists \vec{v} < c\, \varphi(\vec{v}, a).$$

- Now, let

$$\Phi(x) = \{\exists \vec{v} \varphi(\vec{v}, a) \to \exists \vec{v} < x\, \varphi(\vec{v}, a) : \varphi(\vec{v}, u) \in \Pi_{n-1}\}.$$

  This is a recursive type consisting only of Π_n formulas, and is clearly finitely satisfiable.

- Therefore, there exists c that realizes $\Phi(x)$. Therefore, by the last lemma, there exists an initial segment $\mathfrak{A}'$ of $\mathfrak{A}$ which satisfies the conditions of the theorem. □

- The essence of this theorem is that a countable non-standard model of $\mathrm{I}\Sigma_1$ has an initial segment that is isomorphic to itself.
- Friedman first proved this theorem for a countable non-standard model of Peano arithmetic, and several researchers sophisticated it to the above form.
- The same theorem does not hold for non-countable models, and also it does not hold in general for countable non-standard models of $\mathrm{I}\Sigma_0$.
- Furthermore, an important result related to this is McAloon's theorem, which states that a countable non-standard model of $\mathrm{I}\Sigma_0$ has an initial segment that is a model of Peano arithmetic PA.

Thank you for your attention!