# *Topics in Applied Math:*
# Logic and Foundations of Mathematics

## Part 5. Models of first-order arithmetic

**Kazuyuki Tanaka**

BIMSA

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清華大學 求真书院
Qiuzhen College, Tsinghua University

## Logic and Foundations

- **Part 1. Equational theory**
- **Part 2. First order theory**
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- **Part 4. First order arithmetic and incompleteness theorems**
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## Part 5. Schedule

- Nov. 5, (0) Nonstandard models and Overspill
- Nov. 7, (1) The omitting type theorem
- Nov.12, (2) Recursively saturated models
- Nov.14, (3) Friedman's theorem
- Nov.19, (4) Resplendency

# Part 5, §1. Non-standard models (Recap)

- A non-standard model of $\mathrm{I}\Sigma_0$ cannot be recursive, even if it is countable (Tennenbaum's theorem). However, its ordered structure is relatively simple.
- Let $\mathfrak{A}$ be a model of $\mathrm{I}\Sigma_0$. Then, $<$ is a discrete linear order on $A$.
- $\mathfrak{A}$ includes the standard part $\mathfrak{N}$ as an initial segment, i.e., any **non-standard** element $a \in A$ is greater than any standard numbers.
- Now, for two elements $a$ and $b$ of $\mathfrak{A}$, we define $a \sim b$ if $|a - b| \in \mathbb{N}$. The equivalence class $[a]_\sim$ of the non-standard element $a$ is the set of elements represented by $a \pm n$ $(n \in \mathbb{N})$. Therefore, its order type is isomorphic to the order of integers.
- To sum up, the order type of $\mathfrak{A}$ is illustrated as follows.

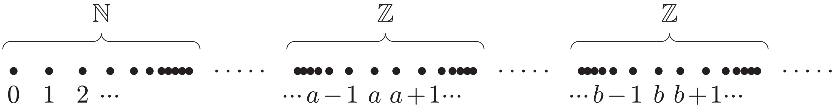

Figure: The order type of a non-standard model of arithmetic

## Theorem 1.1

- The order type of a non-standard model of PA$^-$ is $\mathbb{N} + \mathbb{Z} \bullet \eta$, where $\eta$ is a linear ordering without a maximal element.
- The order type of a non-standard model of $\mathrm{I}\Sigma_0$ is $\mathbb{N} + \mathbb{Z} \bullet \eta$, where $\eta$ is a dense linear order. In particular, the order type of a countable non-standard model of $\mathrm{I}\Sigma_0$ is $\mathbb{N} + \mathbb{Z} \bullet \mathbb{Q}$.

**Example 1**

Let $\mathbb{Z}[X]$ be the set of polynomials in variable $X$ with integer coefficients.
For $p \in \mathbb{Z}[X]$, we set $p > 0$ if its highest order coefficient is positive. Define the order $p > q$ by $p - q > 0$. Then $\mathbb{Z}[X]^+ := \{p \in \mathbb{Z}[X] : p \geq 0\}$ is a non-standard model of PA$^-$, but not a model of $\mathrm{I}\Sigma_0$.

## Lemma 1.2

Let $n \geq 0$. In any non-standard model $\mathfrak{A}$ of $\mathrm{I}\Sigma_n$, $\mathbb{N}$ cannot be defined by a $\Sigma_n$ formula. That is, there is no $\Sigma_n$ formula $\varphi(x)$ such that $i \in \mathbb{N} \Leftrightarrow \mathfrak{A}_A \models \varphi(i)$.

## Theorem 1.3 (Overspill principle)

Let $n \geq 0$ and $\mathfrak{A}$ be any non-standard model of $\mathrm{I}\Sigma_n$, and $\varphi(x)$ be any $\Sigma_n$ formula. If $\mathfrak{A}_A \models \varphi(i)$ holds for infinitely many $i \in \mathbb{N}$, then there exists a non-standard element $a$ such that $\mathfrak{A}_A \models \varphi(a)$ holds.

**Proof.** By way of contradiction, assume $\mathfrak{A}_A \models \neg\varphi(a)$ for any non-standard element $a \in A$. Take any non-standard element $b \in A$. Then a $\Sigma_n$ formula $\psi(x) \ \equiv \ \exists y < b \ (x < y \ \wedge \ \varphi(y))$ defines $\mathbb{N}$, which contradicts with the above lemma. □

It is easy to generalize the above theorem as follows. Let $\mathfrak{A}$ be any non-standard model of $\mathrm{I}\Sigma_n$, and $\varphi(x)$ be any $\Sigma_n$ formula. If a proper subset $S$ of the domain $A$ is closed under $x+1$ and it contains cofinally many elements satisfying $\varphi(x)$, then there exists an element outside of $S$ also satisfying $\varphi(x)$.

# §5.2. The omitting type theorem

- From now, we will consider how to build a new model by extending a given non-standard model without inserting a new element into the original part.

- To extend a given structure by using the completeness theorem, it is sufficient to show that the elementary diagram of the structure is consistent with the existence of a desired element satisfying some condition (which is called "type" here). However, such a method may bring many unexpected elements to the structure.

- The omitting type theorem gives us a method to extend a given structure while omitting elements with a specific condition (type).

## Definition 2.1

- Let $\mathcal{L}$ be any language.
- A set $\Phi(\vec{x})$ of $\mathcal{L}$-formulas that have no free variables other than the $n$ variables $\vec{x} = (x_1, \cdots, x_n)$ is called an **$n$-type**, or simply a **type**.
- If $n$ elements $\vec{a} = (a_1, \cdots, a_n)$ of $\mathcal{L}$-structure $\mathfrak{A}$ satisfies all formulas $\varphi(\vec{x})$ in $\Phi(\vec{x})$ (i.e., $\mathfrak{A}_A \models \varphi(\vec{a})$), we say that $\mathfrak{A}$ **realizes** $\Phi(\vec{x})$ by $\vec{a}$.
- If $\mathfrak{A}$ does not realize $\Phi(\vec{x})$ by any $\vec{a}$, we say that $\mathfrak{A}$ **omits** $\Phi(\vec{x})$.

## Definition 2.2

- Let $T$ be a theory in language $\mathcal{L}$.
- A type $\Phi(\vec{x})$ is called a **type of theory $T$** if $T \cup \Phi(\vec{c})$($\vec{c}$ are new constants) is consistent. That is, there exists a model of $T$ that realizes $\Phi(\vec{x})$.
- Let $\mathfrak{A}$ be an $\mathcal{L}$-structure, and $C$ a subset of the universe of $\mathfrak{A}$. A **type over $C$ in a structure $\mathfrak{A}$** is a type of theory $\mathrm{Th}(\mathfrak{A}_C)$ in language $\mathcal{L}_C$. A type over $C = \varnothing$ is simply called a type.

## Lemma 2.3

Let $\mathfrak{A}$ be an $\mathcal{L}$-structure, and $C$ a subset of $|\mathfrak{A}|$. The following conditions are equivalent.

(1) $\Phi(\vec{x})$ is a type over $C$ in a structure $\mathfrak{A}$.

(2) $\Phi(\vec{x})$ is a type in the language $\mathcal{L}_C$, and the following condition (**finite satisfiability**) holds: For any finite number of $\varphi_1(\vec{x}), \cdots, \varphi_k(\vec{x}) \in \Phi(\vec{x})$,

$$\mathfrak{A}_C \models \exists \vec{x}(\varphi_1(\vec{x}) \wedge \cdots \wedge \varphi_k(\vec{x})).$$

**Proof.**

- Assuming (1). Let $\Phi(\vec{x})$ is a type of theory $\mathrm{Th}(\mathfrak{A}_C)$ in language $\mathcal{L}_C$, and so there exists a model $\mathfrak{B}$ of $\mathrm{Th}(\mathfrak{A}_C)$ that realizes $\Phi(\vec{x})$. Obviously, the finite satisfiability of $\Phi(\vec{x})$ holds in $\mathfrak{B}$, and since $\mathfrak{A}_C$ and $\mathfrak{B}$ are elementary equivalent, $\Phi(\vec{x})$ has also finite satisfiability in $\mathfrak{A}_C$. Therefore, (2) holds.
- Next, assume (2). By the compactness theorem, $\mathrm{Th}(\mathfrak{A}_C) \cup \Phi(\vec{b})$ is consistent and has a model $\mathfrak{B}$. $\mathfrak{B}$ is a model of $\mathrm{Th}(\mathfrak{A}_C)$ and realizes $\Phi(\vec{x})$. Therefore, $\Phi(\vec{x})$ is a type of theory $\mathrm{Th}(\mathfrak{A}_C)$ in the language $\mathcal{L}_C$. That is, (1) holds. □

**Example 3**

- $\Phi_1(x) = \{\exists y(x = y + y),\ \overline{2} < x,\ x < \overline{5}\}$ and $\Phi_2(x) = \{\overline{n} < x : n \in \mathbb{N}\}$ are types (on $C = \varnothing$) in the standard structure of arithmetic $\mathfrak{N}$.
- $\mathfrak{N}$ realizes $\Phi_1(x)$ by $4$, but omits $\Phi_2(x)$.
- However, since any non-standard model of Peano arithmetic PA realizes $\Phi_2(x)$ by an infinite element, $\Phi_2(x)$ is a type of PA.

## Definition 2.4

- A type $\Phi(\vec{x})$ in $\mathcal{L}$ is called a **principal** type of theory $T$, if there exists a formula $\psi(\vec{x})$ in $\mathcal{L}$ such that $T \cup \{\exists\vec{x}\psi(\vec{x})\}$ is consistent, and for any $\varphi(\vec{x}) \in \Phi(\vec{x})$,
$$T \vdash \forall\vec{x}(\psi(\vec{x}) \to \varphi(\vec{x})).$$
- In this case, we say that $\psi(\vec{x})$ **generates** $\Phi(\vec{x})$ in $T$.
- A **non-principal** type of $T$ is a type of $T$ but not principal.
- A type $\Phi(\vec{x})$ over $C(\subseteq A)$ in an $\mathcal{L}$-structure $\mathfrak{A}$ (i.e., a type of theory $\mathrm{Th}(\mathfrak{A}_C)$ in $\mathcal{L}_C$) is a **principal** type, if it is a principal type of theory $\mathrm{Th}(\mathfrak{A}_A)$ in language $\mathcal{L}_A$.

Any $\mathcal{L}$-structure $\mathfrak{A}$ realizes any principal type $\Phi(\vec{x})$ of it. $(\because)$ If $\psi(\vec{x})$ generates $\Phi(\vec{x})$, then by definition $\mathrm{Th}(\mathfrak{A}_A) \cup \{\exists\vec{x}\psi(\vec{x})\}$ is consistent. Since $\mathrm{Th}(\mathfrak{A}_A)$ is a complete theory, it includes $\exists\vec{x}\psi(\vec{x})$ and so $\mathfrak{A}_A \models \exists\vec{x}\psi(\vec{x})$. Therefore, $\Phi(\vec{x})$ is also realized in $\mathfrak{A}$.

**Example 4**

Since $\Phi(x) = \{\overline{n} < x : n \in \mathbb{N}\}$ is omitted by the standard model $\mathfrak{N}$, it is a non-principal type in $\mathfrak{N}$. On the other hand, in a non-standard model, $\psi(x) \equiv x > a$ generates $\Phi(x)$ if $a$ is any infinite element, so $\Phi(x)$ is its principal type.

We now prove that there is a model of $T$ that omits any non-principal type of $T$.

## Theorem 2.5 (The omitting type theorem)

Let $\mathcal{L}$ be a countable language and $T$ be a consistent theory in a language $\mathcal{L}$. Given countably many non-principal types $\Phi_i(x_1, \cdots, x_{n_i})$ of $T$ ($i \in \mathbb{N}$), then there is a countable model of $T$ that omits all $\Phi_i$.

**Proof.** Let $T$ be a consistent theory in a countable language $\mathcal{L}$, and $\Phi_i(x_1, \cdots, x_{n_i})$ ($i \in \mathbb{N}$) be its non-principal types. By modifying Henkin's proof of Gödel's completeness theorem, we will construct a countable model of $T$ that omits all $\Phi_i$. That is, we will build a complete Henkin expansion $T_\omega$ of $T$ with the following property: for a countable set $C$ of Henkin constants (new constants not in $\mathcal{L}$),

$$\forall i\ \forall \vec{\mathsf{c}} \in C^{n_i}\ \exists \varphi(\vec{x}) \in \Phi_i(\vec{x})\ \neg\varphi(\vec{\mathsf{c}}) \in T_\omega. \qquad (※)$$

By the proof of the completeness theorem, if we define a countable structure $\mathfrak{A}$ from $C$,

$$\forall i\ \neg\exists \vec{a} \in A^{n_i}\ \forall \varphi(\vec{x}) \in \Phi_i(\vec{x})\ \mathfrak{A}_A \models \varphi(\vec{a}),$$

which means that $\mathfrak{A}$ omits all $\Phi_i$.

- $T_\omega$ will be obtained as the limit of a countable sequence $T = T_0 \subseteq T_1 \subseteq \cdots$ of consistent theories. To define $T_m$ $(m = 0, 1, \cdots)$, we first enumerate all the sentences in the language $\mathcal{L}_C = \mathcal{L} \cup C$ and denote it as $\{\sigma_m\}$. Then, also enumerate $\{(i, \vec{\mathtt{c}}) : i \in \mathbb{N},\ \vec{\mathtt{c}} = (\mathtt{c}_1, \cdots, \mathtt{c}_{n_i}) \in C^{n_i}\}$ as $\{\gamma_m\}$. Note: $(i, \vec{\mathtt{c}})$ expresses $\Phi_i(\vec{\mathtt{c}})$.
- Using these two infinite sequences $\{\sigma_m\}$ and $\{\gamma_m\}$, we inductively define $T_m$.
- Now, suppose that $T_m$ is defined, and $T_{m+1}$ is obtained from $T_m$ by adding at most three sentences according to the following conditions:

(1) If $T_m \cup \{\sigma_m\}$ is consistent, let $T'_{m+1} = T_m \cup \{\sigma_m\}$, otherwise let $T'_{m+1} = T_m$.

(2) If $T_m \cup \{\sigma_m\}$ is consistent and $\sigma_m$ is in the form $\exists x \theta(x)$, then let $T''_{m+1} = T'_{m+1} \cup \{\theta(\mathtt{d})\}$ with an appropriate Henkin constant $\mathtt{d} \in C$ (not appearing in $T'_{m+1}$). Otherwise, let $T''_{m+1} = T'_{m+1}$.

(3) For $\gamma_m = (i, \vec{\mathtt{c}})$, choose $\varphi(\vec{x}) \in \Phi_i(\vec{x})$ such that $T''_{m+1} \cup \{\neg\varphi(\vec{\mathtt{c}})\}$ is consistent, and let $T_{m+1} = T''_{m+1} \cup \{\neg\varphi(\vec{\mathtt{c}})\}$.

(1) and (2) are used in the proof of completeness theorem to make $T_\omega$ a complete Henkin extension of $T$. If (3) holds, it is clear that $T_\omega$ satisfies the desired property (※), so the rest is to show we can obtain $\varphi(\vec{x}) \in \Phi_i(\vec{x})$ satisfying (3).

(3) Let $\gamma_m = (i, \vec{\mathrm{c}})$. There exists $\varphi(\vec{x}) \in \Phi_i(\vec{x})$ s.t. $T''_{m+1} \cup \{\neg\varphi(\vec{\mathrm{c}})\}$ is consistent.

- To show (3) by contradiction, suppose that for every $\varphi(\vec{x}) \in \Phi_i(\vec{x})$, $T''_{m+1} \cup \{\neg\varphi(\vec{\mathrm{c}})\}$ is inconsistent. Since $T''_{m+1} - T$ is a finite set, we take the conjunction $\wedge$ of all its elements and denote it as $\delta(\vec{\mathrm{c}}, \vec{\mathrm{d}})$. Here, $\vec{\mathrm{d}}$ is a sequence of Henkin constants other than $\vec{\mathrm{c}}$ included in $\delta$.
- Then, for every $\varphi(\vec{x}) \in \Phi_i(\vec{x})$,

$$T \vdash \delta(\vec{\mathrm{c}}, \vec{\mathrm{d}}) \to \varphi(\vec{\mathrm{c}}).$$

- Since $\vec{\mathrm{d}}$ do not appear in $T$ or $\varphi(\vec{\mathrm{c}})$, they can be treated like free variables, and so

$$T \vdash \exists \vec{y}\ \delta(\vec{\mathrm{c}}, \vec{y}) \to \varphi(\vec{\mathrm{c}}).$$

- Since $\vec{c}$ does not appear in $T$,

$$T \vdash \forall \vec{x}(\exists \vec{y}\ \delta(\vec{x}, \vec{y}) \to \varphi(\vec{x})).$$

- Since $\delta(\vec{\mathrm{c}}, \vec{\mathrm{d}})$ is consistent with $T$, $\exists \vec{x} \exists \vec{y}\ \delta(\vec{x}, \vec{y})$ is also consistent with $T$. Since $\exists \vec{y}\ \delta(\vec{x}, \vec{y})$ generates $\Phi_i(\vec{x})$, we get a contradiction with the assumption $\Phi_i(\vec{x})$ is a non-principal type.

$\square$

As an application of the above theorem, we show that any countable model of PA has a proper elementary end-extensions defined below. A similar theorem also holds for set theory.

## Definition 2.6

Let $\mathcal{L}$ be any language containing a binary relational symbol $<$. Let $\mathfrak{A}$, $\mathfrak{B}$ be two $\mathcal{L}$-structures such that $\mathfrak{B}$ is a substructure of $\mathfrak{A}$.
$\mathfrak{A}$ is an **end-extension** of $\mathfrak{B}$, or $\mathfrak{B}$ is an initial segment of $\mathfrak{A}$ if the following holds, which is denoted as $\mathfrak{B} \subseteq_e \mathfrak{A}$.

$$(b \in |\mathfrak{B}| \wedge \mathfrak{A} \models a < b) \Rightarrow a \in |\mathfrak{B}|.$$

If $\mathfrak{B}$ is an elementary substructure of $\mathfrak{A}$, and $\mathfrak{A}$ is an end-extension of $\mathfrak{B}$, then $\mathfrak{A}$ is called an **elementary end-extension** of $\mathfrak{B}$.

In arithmetic, we usually interpret the symbol $<$ as the ordinary $<$ relation.
In set theory, $<$ is interpreted as $\in$ or $\subsetneqq$, but it should be noted that $\in$ does not satisfy the transitivity law.

## Definition 2.7 (c.f. Definition 3.1 in Part 4)

In a language $\mathcal{L}$ with a relation $<$, the following schema is called **collection principle**:

$$\forall x < u \exists y_1 \cdots \exists y_k \varphi(x, y_1, \cdots, y_k) \to \exists v \forall x < u \exists y_1 < v \cdots \exists y_k < v \varphi(x, y_1, \cdots, y_k).$$

where $\varphi(x, y_1, \cdots, y_k)$ is any formula in $\mathcal{L}$, and may include variables other than $v$.

- The fact that the collection principle holds in PA as in the last part.
- In set theory, when $<$ is interpreted as $\in$, it is kind of Fraenkel's axiom (called the "collection principle" or the "replacement axiom"). In fact, the collection principle of arithmetic is an imitation of this Fraenkel axiom.
  Even if we interpret $<$ as $\subsetneq$ in set theory, the collection principle can be proven (in ZF set theory).

### Theorem 2.8

In a countable language $\mathcal{L}$ containing a binary relation symbol $<$, a countably infinite structure that satisfies the collection principle and the transitivity law has a proper elementary end-extension.

**Proof.**

- Let $\mathfrak{A}$ be a countable infinite structure in a countable language $\mathcal{L}$ containing $<$ which satisfies the collection principle and the transitive law.
- First, consider a special case where $\mathfrak{A} \models \forall x \forall y (x \not< y)$. Then, construct a proper elementary extension $\mathfrak{B}$ of $\mathfrak{A}$ by the compactness theorem or any other methods. Since $\mathfrak{B}$ also satisfies $\forall x \forall y (x \not< y)$, it is an end-extension in a trivial sense.
- Next, we assume that there exist two elements $d$ and $e$ such that $\mathfrak{A}_A \models d < e$.
- Then, for a finite number of $a_1, \cdots, a_k \in A$, there is $a_0 \in A$ such that $a_1 < a_0, \cdots, a_k < a_0$. This follows from the collection principle by letting $\varphi(x, y_1, \cdots, y_k)$ be $y_1 = a_1 \wedge \cdots \wedge y_k = a_k$ and $u = e$, $x = d$.

- Let $\mathsf{c}$ be a constant that does not belong to $\mathcal{L}_A$, and $T$ be the following theory in $\mathcal{L}_A \cup \{\mathsf{c}\}$.

$$T = \mathrm{Th}(\mathfrak{A}_A) \cup \{a < \mathsf{c} : a \in A\}.$$

- We can show this theory has a model by the compactness theorem. Any finite subset of $T$ has a model $\mathfrak{A}_A$ with an appropriate interpretation of $\mathsf{c}$, since for any $a_1, \cdots, a_k \in A$, there is $a_0 \in A$ such that $a_1 < a_0, \cdots, a_k < a_0$. Therefore, $T$ itself has a model, which contains an infinite element (an element larger than any $a \in A$) that is an interpretation of $\mathsf{c}$, and it is also an elementary extension of $\mathfrak{A}$.
- However, there is no guarantee that such an expansion becomes an end-extension. To use the omitting type theorem, for each $a \in A$, define a type $\Phi_a$ of $\mathcal{L}_A$ as follows

$$\Phi_a(x) = \{x < a\} \cup \{x \neq b : b < a\}.$$

- We want to show that they are non-principal types of $T$. By way of contradition, we assume that for some $a \in A$, there is a formula $\psi(x, \mathsf{c})$ in $\mathcal{L}_A \cup \{\mathsf{c}\}$ that generates $\Phi_a(x)$. (Note that $T \cup \{\exists x \psi(x, \mathsf{c})\}$ is considered to be consistent).

- Take arbitrary $b < a$. Since
$$T \vdash \psi(x, \mathsf{c}) \to x \neq b,$$
letting $x = b$, we have
$$T \vdash \neg\psi(b, \mathsf{c}).$$
- Since by the definition of $T$, there exist finitely many $a_1, \cdots, a_k \in A$,
$$\mathrm{Th}(\mathfrak{A}_A) \vdash (a_1 < \mathsf{c} \wedge \cdots \wedge a_k < \mathsf{c}) \to \neg\psi(b, \mathsf{c}).$$
- Since $\mathsf{c}$ does not appear in $\mathrm{Th}(\mathfrak{A}_A)$, it can be treated as a variable, and so
$$\mathrm{Th}(\mathfrak{A}_A) \vdash \forall y((y > a_1 \wedge \cdots \wedge y > a_k) \to \neg\psi(b, y)).$$
- By collection, take $a_0 \in A$ such that $a_0 > a_1, \cdots, a_0 > a_k$, and then by transitivity,
$$\mathrm{Th}(\mathfrak{A}_A) \vdash \forall y > a_0 \neg\psi(b, y).$$
Therefore,
$$\mathfrak{A}_A \models \exists z \forall y > z \neg\psi(b, y).$$
Since $b < a$ is taken arbitrarily,
$$\mathfrak{A}_A \models \forall x < a \exists z \forall y > z \neg\psi(x, y).$$
Note that we cannot write $\forall x < a \forall y > a_0 \neg\psi(x, y)$ as $a_0$ depends on $b$.

- Again by collection, we obtain $a' \in A$ such that
$$\mathfrak{A}_A \models \forall x < a \exists z < a' \forall y > z \neg\psi(x, y).$$
And by transitivity, we have
$$\mathfrak{A}_A \models \forall x < a \forall y > a' \neg\psi(x, y).$$
- Since $T = \mathrm{Th}(\mathfrak{A}_A) \cup \{a < \mathsf{c} : a \in A\}$,
$$T \vdash \forall x < a \neg\psi(x, \mathsf{c}), \text{ i.e., } T \vdash \forall x(\psi(x, \mathsf{c}) \to x \not< a).$$
- On the other hand, $\psi(x, \mathsf{c})$ generates $\Phi_a(x)$ and $T \vdash \forall x(\psi(x, \mathsf{c}) \to x < a)$, so
$$T \vdash \forall x \neg\psi(x, \mathsf{c}),$$
which conradicts with the assumption that $T \cup \{\exists x \psi(x, \mathsf{c})\}$ is consistent. □

## Corollary 2.9

A countable model of Peano arithmetic PA has a proper elementary end-extension.

- The above corollary can also be extended to non-countable models, which is called the MacDowell-Specker Theorem. For more details, see Kaye's book *Models of Peano arithmetic*.
- The proof of elementary end-extension for ZF set theory can be found in Chang and Keisler's book *Model theory*. It is also known that the results of set theory cannot be extended to non-countable cases.

**Problem 2: HW # 4-2**

Show that if a model $\mathfrak{A}$ of $\mathrm{I}\Sigma_0$ has a proper elementary end-extension, $\mathfrak{A}$ is a model of PA.

# §3. Recursively saturated models

- Next, we want to construct countable structures that realize as many types $\Phi(\vec{x})$ as possible.

- Even if the language is countable (and so the set of formulas is countable), there can be uncountable many types $\Phi(\vec{x})$, and then it is impossible to realize all of them in countable structures.

- This brings us to the notion of "recursive saturated model", which realizes only the recursive types. Using this model, we prove "Friedman's self-embedding theorem," a groundbreaking discovery on countable non-standard models of arithmetic.

- In a countable language, the type $\Phi(\vec{x})$ is said to be recursive if the set of Gödel numbers of its formulas is recursive (computable).

- By an argument similar to Craig's Lemma in last part, the class of types is essentially the same whether they are CE, recursive, or primitive recursive.

## Definition 3.1

Let $\mathcal{L}$ be a countable language. An $\mathcal{L}$-structure $\mathfrak{A}$ is **recursively saturated** if any recursive 1-type over a finite set $\{a_1, \cdots, a_n\} \subseteq A$ is realized in $\mathfrak{A}$, that is, any recursive type $\Phi(x_0, x_1, \cdots, x_n) = \{\varphi_i(x_0, x_1, \cdots, x_n) \mid i \in \mathbb{N}\}$ and for any $a_1, \cdots, a_n \in A$,

$$\forall j \exists a \in A \forall i < j \, \mathfrak{A}_A \models \varphi_i(a, a_1, \cdots, a_n) \Rightarrow \exists a \in A \forall i \, \mathfrak{A}_A \models \varphi_i(a, a_1, \cdots, a_n).$$

Problem 3

Show that any finite structure is recursively saturated.

- The standard structure of arithmetic $\mathfrak{N}$ is clearly not recursively saturated. However, by the next lemma, there exists a recursively saturated countable non-standard model that is elementary equivalent to $\mathfrak{N}$.

## Lemma 3.2

A countable structure in a countable language has a countable elementary extension which is recursively saturated.

**Proof.**

- Let $\mathfrak{A}$ be a countable structure in a countable language. For each recursive type $\Phi = \{\varphi_i(x_0, x_1, \cdots, x_n) \mid i \in \mathbb{N}\}$ and for each $a_1, \cdots, a_n \in A$, we add a new constant $c_{\Phi, a_1, \cdots, a_n}$ to the language, and let

$$T_1 = \mathrm{Th}(\mathfrak{A}_A) \cup \{\exists x \forall i < j \varphi_i(x, a_1, \cdots, a_n) \to \forall i < j \varphi_i(c_{\Phi, a_1, \cdots, a_n}, a_1, \cdots, a_n) : \\ j \in \mathbb{N} \text{ and } c_{\Phi, a_1, \cdots, a_n} \text{is a new constant}\}.$$

- By the compactness theorem and the downward Löwenheim–Skolem Theorem, $T_1$ has a countable model $\mathfrak{A}_1$.
- Then $\mathfrak{A} \prec \mathfrak{A}_1$ and $\mathfrak{A}_1$ realizes all recursive 1-types on any finite subset of $A$ (in $\mathfrak{A}_1$).
- Next, we construct a countable model $\mathfrak{A}_2 \succ \mathfrak{A}_1$ that realizes all recursive 1-types on a finite subset of $A_1$.

- Similarly, we create $\mathfrak{A}_2 \prec \mathfrak{A}_3 \prec \mathfrak{A}_4 \prec \cdots$, and denote $\mathfrak{A}_\infty = \bigcup_k \mathfrak{A}_k$.
- By the elementary chain theorem in part 3, $\mathfrak{A}_\infty$ is an elementary extension of $\mathfrak{A}$ and is also countable.

**Elementary chain theorem**, revisit

Let $\mathfrak{A}_0 \prec \mathfrak{A}_1 \prec \cdots$ be an elementary chain. Let $\mathfrak{A}$ be the union of the elementary chain. Then for each $i$, $\mathfrak{A}_i \prec \mathfrak{A}$.

- To see that $\mathfrak{A}_\infty$ is recursively saturated, we arbitrarily select a finite number of elements from $\mathfrak{A}_\infty$ and consider a recursive type over them.
- It is a type over $A_k$ for a sufficiently large $k$, and is realized by $\mathfrak{A}_{k+1}$, and also by its elementary extension $\mathfrak{A}_\infty$. □

Thank you for your attention!