Topics in Applied Math: Logic and Foundations of Mathematics

Part 4. First order arithmetic

Kazuyuki Tanaka

BIMSA

November 5, 2025

清華大学 求真书院
Qiuzhen College, Tsinghua University

Logic and Foundations

- **Part 1. Equational theory**
- **Part 2. First order theory**
- **Part 3. Basic Model theory**
- **Part 4. First order arithmetic and incompleteness theorems**
- **Part 5. Models of first-order arithmetic**
- **Part 6. Second order arithmetic and reverse mathematics**

Part 4. Schedule

- Oct. 24, (1) Peano arithmetic
- Oct. 29, (2) representation theorems
- Oct. 31, (3) The first incompleteness theorem
- Nov. 5, (4) The second incompleteness theorem

§4. The first incompleteness theorem

We assume that all theories are given in the language $\mathcal{L}_{\mathrm{OR}}$ and contain R, so Σ_1-complete.

Definition 4.1

A theory T is **ω-consistent** $\Leftrightarrow$ for any formula $\varphi(x)$, if $\varphi(\overline{n})$ is provable in T for all $n \in \mathbb{N}$, then $\exists x \neg\varphi(x)$ is not provable in T. A theory is **1-consistent** $\Leftrightarrow$ it is ω-consistent with $\varphi(x)$ restricted to Σ_0 (or Δ_1). A theory is **Σ_n-sound** if all the Σ_n theorems of T are true.

Homework # 3-2

(1) Show that a Σ_1-complete theory T is 1-consistent iff it is Σ_1-sound.

(2) Show that any ω-consistent Σ_1-complete theory is Π_3-sound, but may not be Σ_3-sound.

Lemma 4.2 ((Weak) Representation Theorem for CE sets)

Suppose that a theory T contains R and 1-consistent. Then, for any CE set C, there exists a Σ_1 formula $\varphi(x)$ such that for any n, $n \in C \ \Leftrightarrow\ T \vdash \varphi(\overline{n})$.

Definition 4.5

Let Ω be a set of symbols with an injection $\phi : \Omega \to \mathbb{N}$. For a string $s = a_0 \cdots a_{n-1} \in \Omega^*$, the following natural number $\ulcorner s \urcorner$ is called the **Gödel number** of s,

$$\ulcorner s \urcorner = p(0)^{\phi(a_0)+1} \cdot p(1)^{\phi(a_1)+1} \cdot \cdots \cdot p(n-1)^{\phi(a_{n-1})+1}.$$

The mapping $\ulcorner \quad \urcorner$ is an injection from the set of all symbols Ω^* to $\mathbb{N}$.

Theorem 4.7 (Craig's lemma)

For any CE theory T, there exists a prim. rec. theory T' that proves the same theorems.

Definition 4.8

We define the primitive recursive predicate Proof_T as follows.

$$\mathrm{Proof}_T(\ulcorner P \urcorner, \ulcorner \sigma \urcorner) \Leftrightarrow P \text{ is a proof of formula } \sigma \text{ in } T'.$$

By Proof_T, we also denote a Δ_1 formula expressing the above Proof_T in $\mathrm{I}\Sigma_1$.
A Σ_1 formula Bew_T is defined as

$$\mathrm{Bew}_T(x) \equiv \exists y \ \mathrm{Proof}_T(y, x).$$

Lemma 4.9 (Diagonalization lemma)

Let T be a theory containing R. For any formula $\psi(x)$ in which x is the unique free variable, there exists a sentence σ such that $T \vdash$ "$\sigma \leftrightarrow \psi(\overline{\ulcorner \sigma \urcorner})$" .

Theorem 4.10 (**Gödel's first incompleteness theorem, a formal version**)

Let T be a 1-consistent Σ_1 theory containing R. Then, T is incomplete, that is, there is a sentence σ such that $T \nvdash \sigma$ and $T \nvdash \neg\sigma$.

Proof.

- By the diagonalization lemma, $\neg\mathrm{Bew}_T(x)$ has a fixed point, that is, there exists σ such that $T \vdash \sigma \leftrightarrow \neg\mathrm{Bew}_T(\overline{\ulcorner \sigma \urcorner})$.
- In T, σ is neither provable (by Σ_1 completeness) nor disprovable (by 1-consistency).

The sentence σ asserts its own unprovability, and it is called the **Gödel sentence** of T. The Gödel sentence is a true Π_1 sentence.

To weaken the assumption of incompleteness theorem, Rosser modified $\mathrm{Bew}_T(x)$ as follows

$$\mathrm{Bew}^*_T(\overline{\ulcorner\sigma\urcorner}) \equiv \exists y\ (\mathrm{Proof}_T(y, \overline{\ulcorner\sigma\urcorner}) \land \forall z < y\ \neg\mathrm{Proof}_T(z, \overline{\ulcorner\neg\sigma\urcorner})).$$

Lemma 4.11

Let T be a Σ_1 theory containing R. Then, for any sentence σ,
(1) $T \vdash \sigma \Rightarrow T \vdash \mathrm{Bew}^*_T(\overline{\ulcorner\sigma\urcorner})$,
(2) $T \vdash \neg\sigma \Rightarrow T \vdash \neg\mathrm{Bew}^*_T(\overline{\ulcorner\sigma\urcorner})$.

Proof. We may assume T is consistent. Then, (1) follows from Σ_1 completeness. To show (2), assume $T \vdash \neg\sigma$. There exists $n \in \mathbb{N}$ such that the following holds in $\mathfrak{N}$

$$\mathrm{Proof}_T(\overline{n}, \overline{\ulcorner\neg\sigma\urcorner}) \land \forall z \leq \overline{n} \neg\mathrm{Proof}_T(z, \overline{\ulcorner\sigma\urcorner}).$$

By Σ_1 completeness, the above formula is provable in T. So, since T contains R, T proves

$$\forall y(\mathrm{Proof}_T(y, \overline{\ulcorner\sigma\urcorner}) \to \exists z < y \mathrm{Proof}_T(z, \overline{\ulcorner\neg\sigma\urcorner})), \text{ i.e., } \neg\mathrm{Bew}^*_T(\overline{\ulcorner\sigma\urcorner}). \qquad \square$$

Theorem 4.12 (Gödel-Rosser)

Let T be a consistent Σ_1 theory containing R, and σ a fixed point of $\neg\mathrm{Bew}^*_T(x)$. Then $T \nvdash \sigma$ and $T \nvdash \neg\sigma$.

Two applications of the first incompleteness thm

Theorem 4.14 (Tarski's undefinability of truth)

There is no formula $\psi(x)$ such that $\mathfrak{N} \models \sigma \leftrightarrow \psi(\overline{\ulcorner \sigma \urcorner})$ for all sentence σ.

Lemma 4.16

For a consistent Σ_1 theory $T \supset \mathsf{R}$, $\{\ulcorner \sigma \urcorner : T \vdash \sigma, \sigma \text{ is a sentence}\}$ is not computable.

Proof. By the strong representation theorem and the diagonalization lemma.

Theorem 4.17 (Undecidability of first-order logic, due to Church-Turing)

$\{\ulcorner \sigma \urcorner : \ \sigma \text{ is a valid sentence in the language} \mathcal{L}_{\mathrm{OR}}\}$ is not computable. Therefore, the satisfiability of first order logic is not decidable.

Proof. Let ξ be a sentence obtained by connecting all the axioms of $\mathsf{Q}_<$ by $\land$. If $\{\ulcorner \sigma \urcorner : \vdash \sigma\}$ is computable, $\{\ulcorner \sigma \urcorner : \vdash \xi \to \sigma\} = \{\ulcorner \sigma \urcorner : \mathsf{Q}_< \vdash \sigma\}$ is also computable, which contradicts with the above lemma.
Finally, the satisfiability of first order logic can be expressed as $\{\ulcorner \sigma \urcorner : \nvdash \neg\sigma\}$. □

§5. Intoducing the second incompleteness theorem

- The first incompleteness theorem says that a consitent CE theory T containing R is neither prove nor disprove the Gödel sentence.
- The second incompleteness theorem says that a consistent CE theory T containing $\mathsf{I}\Sigma_1$ does not prove its consistency.
- To obtain the second theorem, it is sufficient to show that the consistency implies the Gödel sentence, or equivalently the consistency implies the unprovability of the Gödel sentence.
- Thus, the main part of the proof of the second theorem is to formalize the proof of the first theorem in the system T.
- Although this requires extremely elaborate arguments, the main points are summarized as the three properties of the derivability predicate $\mathrm{Bew}_T(x)$ as shown in the next slide.

Lemma 5.1 (**Hilbert-Bernays-Löb's derivability condition**)

Let T be a consistent CE theory containing $\mathrm{I}\Sigma_1$. For any φ, ψ,
D1. $T \vdash \varphi \Rightarrow T \vdash \mathrm{Bew}_T(\overline{\ulcorner \varphi \urcorner})$.
D2. $T \vdash \mathrm{Bew}_T(\overline{\ulcorner \varphi \urcorner}) \land \mathrm{Bew}_T(\overline{\ulcorner \varphi \to \psi \urcorner}) \to \mathrm{Bew}_T(\overline{\ulcorner \psi \urcorner})$.
D3. $T \vdash \mathrm{Bew}_T(\overline{\ulcorner \varphi \urcorner}) \to \mathrm{Bew}_T(\overline{\ulcorner \mathrm{Bew}_T(\overline{\ulcorner \varphi \urcorner}) \urcorner})$.

Proof.

- D1 is obtained from the Σ_1 completeness of T, since $\mathrm{Bew}_T(\overline{\ulcorner \varphi \urcorner})$ is a Σ_1 formula.
- For D2, it is clear that the proof of ψ is obtained by applying MP to the proof of φ and the proof of $\varphi \to \psi$.
- D3 formalizes D1 in T. This is the most difficult, since we can not find a simple machinery to transform a proof of φ in T to a proof of $\mathrm{Bew}_T(\overline{\ulcorner \varphi \urcorner})$. There are several known ways to deal with this problem, but below we will briefly explain how to deal with the representability of primitive recursive functions within the system.

- Since the function from a number n to the Gödel number of its numeral $\ulcorner \overline{n} \urcorner$ is primitive recursive, we denote the function by $\dot{x}$.
- For an expression $\varphi(x)$, $\varphi(\dot{y})$ denotes the expression obtained by substituting the term with the Gödel number $\dot{y}$ to every free occurrence of the variable x. If the value of y is a standard natural number n, this is nothing but a substitution of the numeral $\overline{n}$, but $\varphi(\dot{y})$ is just an expression with the variable y, which can be formulated within Bew_T.
- With this notation, our goal is to prove

$$T \vdash \mathrm{Proof}_T(x, y) \to \mathrm{Bew}_T(\overline{\ulcorner \mathrm{Proof}_T(\dot{x}, \dot{y}) \urcorner}). \tag{1}$$

- In general, we prove that for any primitive recursive function f,

$$T \vdash f(x_1, \ldots, x_k) = y \to \mathrm{Bew}_T(\overline{\ulcorner f(\dot{x_1}, \ldots, \dot{x_k}) = \dot{y}) \urcorner}). \tag{2}$$

- The above formula can be proved by meta-induction on the construction of the primitive recursive function f.
- As a example, we will prove for addition $x + y = z$, the above formula (2) holds.

- By Σ_1 induction on variable y (assuming other variables are constants), we prove that
$$x + y = z \to \mathrm{Bew}_T(\overline{\ulcorner \dot{x} + \dot{y} = \dot{z}) \urcorner}). \tag{3}$$
- First, if $y = 0$, then $x + 0 = z$ and so $x = z$. By A3 of PA, $\mathrm{Bew}_T(\overline{\ulcorner \dot{x} + 0 = \dot{x}) \urcorner})$. Thus
$$x + 0 = z \to \mathrm{Bew}_T(\overline{\ulcorner \dot{x} + 0 = \dot{z}) \urcorner}).$$
- Next assuming $x + y = w \to \mathrm{Bew}_T(\overline{\ulcorner \dot{x} + \dot{y} = \dot{w}) \urcorner})$, we want to show
$$x + (y + 1) = z \to \mathrm{Bew}_T(\overline{\ulcorner \dot{x} + (\dot{y} + 1) = \dot{z}) \urcorner}).$$
- Suppose $x + (y + 1) = z$. Let $w = x + y$. Then, we have $z = w + 1$.
- By the induction hypothesis, $\mathrm{Bew}_T(\overline{\ulcorner \dot{x} + \dot{y} = \dot{w} \urcorner})$. So, by using A4 in Bew_T, $\mathrm{Bew}_T(\overline{\ulcorner \dot{x} + (\dot{y} + 1) = \dot{w} + 1) \urcorner})$. Then from $\mathrm{Bew}_T(\overline{\ulcorner \dot{z} = \dot{w} + 1 \urcorner})$, we obtain $\mathrm{Bew}_T(\overline{\ulcorner \dot{x} + (\dot{y} + 1) = \dot{z}) \urcorner})$.
- Thus, we have shown (3) by IΣ_1.
- As for other p.r. functions, their defining formulas are given as axioms in the theory T, so (2) can be proved using a similar argument.

- To prove D3, assume $\mathrm{Bew}_T(\overline{\ulcorner \varphi \urcorner})$ in addition to T. Then, there is a numeral c that satisfies $\mathrm{Proof}_T(c, \overline{\ulcorner \varphi \urcorner})$.
- Now by (1), we have $\mathrm{Bew}_T(\overline{\ulcorner \mathrm{Proof}_T(\dot{c}, \ulcorner \dot{\varphi} \urcorner) \urcorner})$. Here, $\ulcorner \dot{\varphi} \urcorner$ is a standard natural number, so it is nothing but $\overline{\ulcorner \varphi \urcorner}$.
- Since $T \vdash \mathrm{Proof}_T(\dot{c}, \overline{\ulcorner \varphi \urcorner}) \to \exists x \mathrm{Proof}_T(x, \overline{\ulcorner \varphi \urcorner})$ can be deduced from a quantification axiom of first-order logic, we have

$$T \vdash \mathrm{Proof}_T(\dot{c}, \overline{\ulcorner \varphi \urcorner}) \to \mathrm{Bew}_T(\overline{\ulcorner \varphi \urcorner}).$$

- Then, by D1,

$$T \vdash \mathrm{Bew}_T\left(\overline{\ulcorner \mathrm{Proof}_T(\dot{c}, \overline{\ulcorner \varphi \urcorner}) \to \mathrm{Bew}_T(\overline{\ulcorner \varphi \urcorner}) \urcorner}\right).$$

  By D2,

$$T \vdash \mathrm{Bew}_T\left(\overline{\ulcorner \mathrm{Proof}_T(\dot{c}, \overline{\ulcorner \varphi \urcorner})) \urcorner}\right) \to \mathrm{Bew}_T\left(\overline{\ulcorner \mathrm{Bew}_T(\overline{\ulcorner \varphi \urcorner}) \urcorner}\right).$$

- Finally, $\mathrm{Bew}_T(\overline{\ulcorner \mathrm{Bew}_T(\overline{\ulcorner \varphi \urcorner}) \urcorner})$ is obtained with the first assumption by MP. Thus, D3 is proven.

□

In the following, let π_G denote the Gödel sentence in the proof of the first incompleteness theorem. That is,

$$T \vdash \pi_G \leftrightarrow \neg \mathrm{Bew}_T(\overline{\ulcorner \pi_G \urcorner}).$$

By $\mathrm{Con}(T)$, we denote the sentence meaning "T is consistent", formally defined as

$$\mathrm{Con}(T) \equiv \neg \mathrm{Bew}_T(\overline{\ulcorner 0 = 1 \urcorner}).$$

Then we have the following.

Lemma 5.2

$T \vdash \mathrm{Con}(T) \leftrightarrow \pi_G$.

Proof. • To show $\pi_G \to \mathrm{Con}(T)$.

Obviously, $T \vdash 0 = 1 \to \pi_G$. So, by D1 and D2,

$$T \vdash \mathrm{Bew}_T(\overline{\ulcorner 0 = 1 \urcorner}) \to \mathrm{Bew}_T(\overline{\ulcorner \pi_G \urcorner}).$$

Taking the contraposition, we have $T \vdash \pi_G \to \mathrm{Con}(T)$.

- To show $\mathrm{Con}(T) \to \pi_G$.

First, from $T \vdash \pi_G \leftrightarrow \neg \mathrm{Bew}_T(\overline{\ulcorner \pi_G \urcorner})$ and D1,

$$T \vdash \mathrm{Bew}_T(\overline{\ulcorner \mathrm{Bew}_T(\overline{\ulcorner \pi_G \urcorner}) \to \neg \pi_G \urcorner}).$$

Using D2, we have

$$T \vdash \mathrm{Bew}_T(\overline{\ulcorner \mathrm{Bew}_T(\overline{\ulcorner \pi_G \urcorner}) \urcorner}) \to \mathrm{Bew}_T(\overline{\ulcorner \neg \pi_G \urcorner}).$$

Combining this with D3: $T \vdash \mathrm{Bew}_T(\overline{\ulcorner \pi_G \urcorner}) \to \mathrm{Bew}_T(\overline{\ulcorner \mathrm{Bew}_T(\overline{\ulcorner \pi_G \urcorner}) \urcorner})$, we obtain

$$T \vdash \mathrm{Bew}_T(\overline{\ulcorner \pi_G \urcorner}) \to \mathrm{Bew}_T(\overline{\ulcorner \neg \pi_G \urcorner}).$$

Then, by using $T \vdash \pi_G \to (\neg \pi_G \to 0 = 1)$ and D2, we get

$$T \vdash \mathrm{Bew}_T(\overline{\ulcorner \pi_G \urcorner}) \to \mathrm{Bew}_T(\overline{\ulcorner 0 = 1 \urcorner})$$

Taking the contraposition,

$$T \vdash \neg \mathrm{Bew}_T(\overline{\ulcorner 0 = 1 \urcorner}) \to \neg \mathrm{Bew}_T(\overline{\ulcorner \pi_G \urcorner}).$$

That is, $T \vdash \mathrm{Con}(T) \to \pi_G$. $\square$

Theorem 5.3 (**Gödel's second incompleteness theorem)**

Let T be a consistent CE theory, which contains $\mathrm{I}\Sigma_1$. Then $T \nvdash \mathrm{Con}(T)$.
In other words, T cannot prove its own consistency $\mathrm{Con}(T)$.

Proof

By the proof of the first incompleteness theorem, $T \nvdash \pi_G$.
By the above lemma, $T \vdash \mathrm{Con}(T) \leftrightarrow \pi_G$. So, $T \nvdash \mathrm{Con}(T)$. □

Remark

- The first incompleteness theorem is a negative result in the sense that it shows the limit of provability, whereas the second incompleteness theorem shows that the concrete proposition $\mathrm{Con}(T)$ is not provable in T, which provides a positive tool from an application perspective.
- In mathematical logic, the second incompleteness theorem is often used to separate two axiomatic theories by showing the consistency of one over the other. E.g. $\mathrm{I}\Sigma_1$ is a proper subsystem of PA, since the consistency of the former can be proved in the latter.

Homework # 4-1

(1) Show that there is a consistent theory T that proves its own contradiction $\neg\mathrm{Con}(T)$.

(2) Let $\mathrm{Bew}^{\#}_T(x) \equiv (\mathrm{Bew}_T(x) \land x \neq \overline{\ulcorner 0 = 1 \urcorner})$. For any true proposition σ,

$$\mathrm{Bew}^{\#}_T(\overline{\ulcorner \sigma \urcorner}) \leftrightarrow \mathrm{Bew}_T(\overline{\ulcorner \sigma \urcorner})$$

and

$$T \vdash \neg\mathrm{Bew}^{\#}_T(\overline{\ulcorner 0 = 1 \urcorner}).$$

Does it contradict with the second incompleteness theorem?

- As a variant of the Gödel sentence, a sentence meaning "this sentence is provable" is known as a **Henkin sentence**. That is, H is a Henkin sentence if

  $$\mathrm{H} \leftrightarrow \mathrm{Bew}_T(\overline{\ulcorner \mathrm{H} \urcorner}).$$

  If H is provable and true, then both sides are true and there is no problem.
  On the other hand, if H is false and unprovable, both sides are also equivalent. So, there does not seem to be any clue to determine whether or not H is provable or true. Yet, we can show it is actually provable.
- To this end, first let C denote the sentence "this sentence is consistent with T", i.e., $\mathrm{C} \leftrightarrow \neg\mathrm{Bew}_T(\overline{\ulcorner \neg\mathrm{C} \urcorner})$.
- Since the theory T+C proves its own consistency, it is inconsistent by the second incompleteness theorem. Thus, T proves $\neg\mathrm{C}$.
- On the other hand, since $\neg\mathrm{C} \leftrightarrow \mathrm{Bew}_T(\overline{\ulcorner \neg\mathrm{C} \urcorner})$, $\neg\mathrm{C}$ is the same as H, and therefore H is provable.

The above fact can be also stated as follows.

Theorem 5.4 (Löb's theorem)

Let T be a consistent Σ_1 theory containing $\mathrm{I}\Sigma_1$. If T proves "if T proves σ, then σ", then T proves σ.

Proof.
Suppose that T proves that "if T proves σ, then σ", which means that "If $\neg\sigma$, then T does not prove σ, that is, $T + \neg\sigma$ is consistent." That is, since $T + \neg\sigma$ proves the consistency of $T + \neg\sigma$, by the second incompleteness theorem, $T + \neg\sigma$ is inconsistent. Therefore, T proves σ. □

The Henkin sentence H satisfies that T proves "if T proves H, then H". So by the theorem, T proves H.
A paradoxical fact derived from this theorem is that any proposition σ can be proven by assuming that there is a proof of σ.

Alternative proof of D3

- For simplicity, let T be PA. We also identify a formula $\varphi(x)$ with the set $\{n : \varphi(n)\}$.
- In T, we can prove a countable version of the completeness theorem of first-order logic. A countable model M can be treated as its coded diagram, i.e., the set of the Gödel numbers of $\mathcal{L}_M$-sentences true in M. The arithmetized completeness theorem says that if T' is consistent then there exists (a formula expressing the diagram of) a model of T'.
- Now, we going to prove $\mathrm{Con}(T) \to \pi_G$ in T. By the completeness theorem, it is sufficient to show that any model M of $T + \mathrm{Con}(T)$ satisfies π_G. First, note that π_G is equivalet to $\neg\mathrm{Bew}_T(\overline{\ulcorner \pi_G \urcorner})$, which is also equivalet to $\mathrm{Con}(T + \neg\pi_G)$. Since M satisfies $\mathrm{Con}(T)$, we can make a model M_1 of T over M. So, if M_1 satisfies $\neg\pi_G$, then M shows $\mathrm{Con}(T + \neg\pi_G)$. If M_1 satisfies π_G, M also satisfies π_G since π_G is Π_1 and M is a submodel of M_1. (This proof is due to Kikuchi-Tanaka.)

Some commentaries on Gödel's theorem

- D. Hilbert and P. Bernays, Grundlagen der Mathematik I-II, Springer-Verlag, 1934-1939, 1968-1970 (2nd ed.). This gives the first complete proof of the second incompleteness theorem by analyzing the provability predicate.
- R.M. Smullyan, Theory of Formal Systems, revised edition, Princeton Univ. 1961. A classic masterpiece introducing recursive inseparability, etc.
- Smoryński's chapter on incompleteness theorems in Handbook of Mathematical Logic (1977), edited by J. Barwise
  A wide range of mathematical viewpoints.
- P. Lindström, Aspects of Incompleteness, Lecture Notes in Logic 10, Second edition, Assoc. for Symbolic Logic, A K Peters, 2003.
  A most technically advanced book.
- T. Franzen, Gödel's Theorem: An Incomplete Guide to Its Use and Abuse(2005).
  For a broader understanding of Gödel's theorem.
- P. Smith, Gödel's Without (Too Many) Tears, Second Edition 2022.
  https://www.logicmatters.net/resources/pdfs/GWT2edn.pdf
  Easy to read. The best reference to this lecture.

Thank you for your attention!

Introduction to Part 5, §1. Non-standard models

- We will discuss the general features of non-standard models of arithmetic, especially the properties of their ordered structure.
- A non-standard model of $\mathrm{I}\Sigma_0$ cannot be defined recursively, even if it is countable (Tennenbaum's theorem). However, its ordered structure is relatively simple.
- Let $\mathcal{L}_{\mathrm{OR}}$ structure $\mathfrak{A}$ be a model of $\mathrm{I}\Sigma_0$. Since $\mathfrak{A}$ is also a model of PA$^-$, $<$ is a discrete linear order on A.
- $\mathfrak{A}$ includes the standard part $\mathfrak{N}$ as an initial segment, i.e., any **non-standard** element $a \in A$ is greater than any standard numbers.
- Now, for two elements a and b of $\mathfrak{A}$, we define $a \sim b$ if $|a - b| \in \mathbb{N}$. The equivalence class $[a]_\sim$ of the non-standard element a is the set of elements represented by $a \pm n$ $(n \in \mathbb{N})$. Therefore, its order type is isomorphic to the order of integers.
- To sum up, the order type of $\mathfrak{A}$ is illustrated as follows.

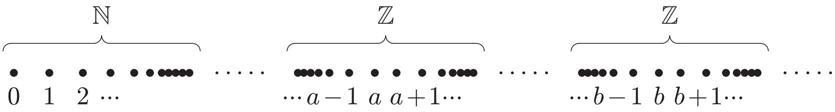

Figure: The order type of a non-standard model of arithmetic

Theorem 5.5

- The order type of a non-standard model of PA$^-$ is $\mathbb{N} + \mathbb{Z} \bullet \eta$, where η is a linear ordering without a maximal element.
- The order type of a non-standard model of $\mathrm{I}\Sigma_0$ is $\mathbb{N} + \mathbb{Z} \bullet \eta$, where η is a dense linear order. In particular, the order type of a countable non-standard model of $\mathrm{I}\Sigma_0$ is $\mathbb{N} + \mathbb{Z} \bullet \mathbb{Q}$.

Proof.

- As explained in the last slide, the order type of a non-standard model of PA$^-$ is $\mathbb{N} + \mathbb{Z} \bullet \eta$, where η is a linear order. Moreover, η has no largest element since $[a]_\sim < [a + a]_\sim$.
- Suppose $\mathfrak{A}$ is a model of $\mathrm{I}\Sigma_0$. If $[a]_\sim < [b]_\sim$, there exists c s.t. $[a]_\sim < [c]_\sim < [b]_\sim$. Because $c \sim (a + b)/2$ can be obtained s.t. $(a + b = 2c) \vee (a + b = 2c + 1)$. (In PA$^-$, this property is not provable.) Thus for any non-standard element b, there exists a non-standard element c such that $[c]_\sim < [b]_\sim$. Therefore, η is a dense linear order with no maximum or minimum.
- According to Cantor's theorem (which will be shown later), any countable dense linear order without end-points is isomorphic to the order type of rational numbers, so when $\mathfrak{A}$ is countable, its order type is uniquely determined as $\mathbb{N} + \mathbb{Z} \bullet \mathbb{Q}$. ☐

Example 1

- Let $\mathbb{Z}[X]$ be the set of polynomials in variable X with integer coefficients. With $+, \bullet, 0, 1$ naturally defined on it, $\mathbb{Z}[X]$ turns out to be a ring.
- For $p \in \mathbb{Z}[X]$, we set $p > 0$ if its highest order coefficient is positive. Then define the order $p > q$ by $p - q > 0$.
- Let
$$\mathbb{Z}[X]^+ = \{p \in \mathbb{Z}[X] : p \geq 0\}.$$
Then $\mathbb{Z}[X]^+$ is a non-standard model of PA$^-$ (check it).
- In $\mathbb{Z}[X]^+$, any non-standard element smaller than X can be written as $X - n$. Thus, there is no element between $[X]_\sim$ and the standard part $\mathbb{N}$.
- In other words, $\mathbb{Z}[X]^+$ cannot be a model of $\mathrm{I}\Sigma_0$.

Lemma 5.6

Let $n \geq 0$. In any non-standard model $\mathfrak{A}$ of $\mathrm{I}\Sigma_n$, $\mathbb{N}$ cannot be defined by a Σ_n formula. That is, there is no Σ_n formula $\varphi(x)$ such that $i \in \mathbb{N} \Leftrightarrow \mathfrak{A}_A \models \varphi(i)$.

Proof. Assume to the contrary that for a non-standard model $\mathfrak{A}$ of $\mathrm{I}\Sigma_n$, there exists a Σ_n formula $\varphi(x)$ that defines $\mathbb{N}$. Then,

$$\mathfrak{A} \models \varphi(0) \land \forall x(\varphi(x) \to \varphi(x+1)).$$

So by induction $\mathfrak{A} \models \forall x \varphi(x)$, and hence $\varphi(x)$ does not define $\mathbb{N}$. ☐

By the above proof, we can also easily see that, in a model $\mathfrak{A}$ of $\mathrm{I}\Sigma_n$, any proper subset that includes 0 and is closed under $x+1$ cannot be defined by a Σ_n formula.

Theorem 5.7 (Overspill principle)

Let $n > 0$ and $\mathfrak{A}$ be any non-standard model of $\mathrm{I}\Sigma_n$, and $\varphi(x)$ be any Σ_n formula. If $\mathfrak{A}_A \models \varphi(i)$ holds for infinitely many $i \in \mathbb{N}$, then there exists a non-standard element a such that $\mathfrak{A}_A \models \varphi(a)$ holds.

Proof. If there is no non-standard element a such that $\mathfrak{A}_A \models \varphi(a)$, then $\mathbb{N}$ can be defiend by the Σ_n formula $\psi(x) \equiv \exists y(x < y \land \varphi(y))$, which contradicts with the above lemma. □

It is easy to have the following generalization. In a proper subset closed under $x + 1$ of a model $\mathfrak{A}$ of $\mathrm{I}\Sigma_n$, if the set of elements satisfying a Σ_n formula $\varphi(x)$ has no upper bound, then there is an element outside of the subset also satisfying $\varphi(x)$.

Example 2

There is no non-standard model of $\mathrm{I}\Sigma_0$ with the order type $\mathbb{N} + \mathbb{Z} \bullet \mathbb{R}$.

- If there is such a model $\mathfrak{A}$, $\{[na]_\sim : n \in \mathbb{N}\}$ has an upper bound $[aa]_\sim$, so due to the nature of $\mathbb{R}$, it has a least upper bound $[b]_\sim$.
- However, if $na < b$ for any $n \in \mathbb{N}$, by the overspill principle, $ca < b$ for some non-standard element c.
- Then, $[(c-1)a] < [b]_\sim$, and since $[(c-1)a]_\sim$ is also an upper bound of $\{[na]_\sim : n \in \mathbb{N}\}$, which contradicts with the minimality of $[b]_\sim$.

Problem 1

Show that there is no non-standard model of PA$^-$ with the order type $\mathbb{N} + \mathbb{Z} \bullet \mathbb{R}$.