

Topics in Applied Math: Logic and Foundations of Mathematics

Part 4. First order arithmetic

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Logic and Foundations

- **Part 1. Equational theory**
- **Part 2. First order theory**
- **Part 3. Basic Model theory**
- **Part 4. First order arithmetic and incompleteness theorems**
- **Part 5. Models of first-order arithmetic**
- **Part 6. Second order arithmetic and reverse mathematics**

Part 4. Schedule

- Oct. 24, (1) Peano arithmetic and representation theorems
- **Oct. 29, (2) The first incompleteness theorem**
- Oct. 31, (3) The second incompleteness theorem

Definition 1.1 (Peano arithmetic)

Peano arithmetic (PA) is a first-order theory in the language $\mathcal{L}_{\text{OR}} = \{+, \cdot, 0, 1, <\}$ consisting of the following axioms.

Successor:	A1. $\neg(x + 1 = 0)$,	A2. $x + 1 = y + 1 \rightarrow x = y$.
Addition:	A3. $x + 0 = x$,	A4. $x + (y + 1) = (x + y) + 1$.
Multiplication:	A5. $x \cdot 0 = 0$,	A6. $x \cdot (y + 1) = x \cdot y + x$.
Inequality	A7. $\neg(x < 0)$,	A8. $x < y + 1 \leftrightarrow x < y \vee x = y$.
Induction:	A9. $\varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(x + 1)) \rightarrow \forall x\varphi(x)$.	

Definition 1.2 (Arithmetical Hierarchy)

A formula in \mathcal{L}_{OR} is said to be **bounded** (Σ_0, Π_0), if it is constructed from atomic formulas by using propositional connectives and bounded quantifiers $\forall x < t$ and $\exists x < t$.

$\forall x_1 \cdots \forall x_k \varphi$ is Π_{i+1} if $\varphi \in \Sigma_i$, and $\exists x_1 \cdots \exists x_k \varphi$ is Σ_{i+1} if $\varphi \in \Pi_i$ ($i, k \in \mathbb{N}$).

Definition 1.3

By $I\Gamma$, we denote a subsystem of PA obtained by restricting ($\varphi(x)$ of) induction to the class Γ of formulas in \mathcal{L}_{OR} . Note: $\Gamma = \text{Open}$ is the set of quantifier-free formulas.

Definition 1.4

Robinson's system Q is obtained from PA by removing the axioms of inequality and induction, and instead adding **A10** (Predecessor axiom): $\forall x(x \neq 0 \rightarrow \exists y(y + 1 = x))$.

Let $Q_{<}$ be the system Q plus axiom **A7.5** $\forall x \forall y(x < y \leftrightarrow \exists z(z + (x + 1) = y))$.

Lemma 1.5

In $I\text{Open}$, the following axioms of **theory of discrete ordered semirings** PA^- are provable.

- (1) Semiring axioms (excluding the additive inverses from the commutative ring).
- (2) difference axiom $x < y \rightarrow \exists z(z + (x + 1) = y)$.
- (3) a linear order with the minimum element 0 and discrete ($0 < x \leftrightarrow 1 \leq x$).
- (4) Order preservation $x < y \rightarrow x + z < y + z \wedge (x \cdot z < y \cdot z \vee z = 0)$.

Corollary 1.6

$Q_{<} \subset PA^- \subset I\text{Open} \subset I\Sigma_0 \subset I\Sigma_1 \subset PA$.

§4.2. A weakest system R

A term $\bar{n} = \overbrace{1 + \cdots + 1}^{n \text{ times}}$ is called the **numeral** of number n . Note $\bar{0} = 0$.

Definition 2.1 (Mostowski-Robinson-Tarski's system R)

R is a theory in the language of ordinal rings, consisting of the following axiom schemes.

- R1. $\bar{m} \neq \bar{n}$ (when $m \neq n$).
 R2. $\neg(x < \bar{0})$.
 R3. $x < \overline{n+1} \leftrightarrow x = \bar{0} \vee \cdots \vee x = \bar{n}$.
 R4. $x < \bar{n} \vee x = \bar{n} \vee \bar{n} < x$.
 R5. $\bar{m} + \bar{n} = \overline{m+n}$.
 R6. $\bar{m} \cdot \bar{n} = \overline{m \cdot n}$.

- Originally, R was formulated in the language with \leq instead of $<$. Thus, R2 and R3 are combined as R2.5: $x \leq \bar{n} \leftrightarrow x = \bar{0} \vee \cdots \vee x = \bar{n}$. R4 is rewritten as $x \leq \bar{n} \vee \bar{n} \leq x$.
- In fact, the original definition for R has R2.5': $x \leq \bar{n} \rightarrow x = \bar{0} \vee \cdots \vee x = \bar{n}$ instead of R2.5, but they are equivalent.
- An interesting system R_0 is obtained from R by removing R4. It is still Σ_1 -complete. If we remove R4 from the original system with R2.5', it is no longer Σ_1 -complete.

Lemma 2.2

$Q_{<}$ proves all axioms of R.

Theorem 2.3 (Σ_1 -completeness of R)

R proves all true Σ_1 sentences.

Proof If a Σ_1 sentence $\exists x_1 \exists x_2 \dots \exists x_k \varphi(x_1, x_2, \dots, x_k)$ is true, there exist natural numbers n_1, n_2, \dots, n_k such that $\varphi(\overline{n_1}, \overline{n_2}, \dots, \overline{n_k})$ holds.

By virtue of R3, $\exists x < t \varphi(x)$ can be rewritten as $\varphi(\overline{0}) \vee \varphi(\overline{1}) \vee \dots \vee \varphi(\overline{n-1})$ if the value of close term t is n . Thus, by induction, a bounded sentence can be rewritten as a Boolean combination of atomic sentences, which can be proved/disproved in R if they are true/false. Therefore, $\varphi(\overline{n_1}, \overline{n_2}, \dots, \overline{n_k})$ is provable since it is true. From the rule of first-order logic, $\exists x_1 \exists x_2 \dots \exists x_k \varphi(x_1, x_2, \dots, x_k)$ is also provable in R. \square

Corollary 2.4

$Q_{<}$, PA^- , $IOpen$, etc. are all Σ_1 -complete.

Definition 2.5 (Representation for Sets and Functions)

- A set $C \subset \mathbb{N}$ is **(strongly) representable** by a formula $\varphi(x)$ in a theory T , if

$$n \in C \Rightarrow T \vdash \varphi(\bar{n}), \quad n \notin C \Rightarrow T \vdash \neg\varphi(\bar{n}).$$

- A number-theoretic function $f(\vec{x})$ is **functionally representable** by a formula $\varphi(\vec{x}, y)$ in a theory T , if $\varphi(\vec{x}, y)$ represents $f(\vec{x}) = y$ and for all natural numbers m_1, \dots, m_l ,

$$T \vdash \forall y \forall y' (\varphi(\bar{m}_1, \dots, \bar{m}_l, y) \wedge \varphi(\bar{m}_1, \dots, \bar{m}_l, y') \rightarrow y = y').$$

Definition 2.6

Let $\mathfrak{N} = (\mathbb{N}, +, \cdot, 0, 1, <)$ be the standard model of PA.

- A set $A \subseteq \mathbb{N}^l$ is said to be Σ_i if there exists a Σ_i formula $\varphi(x_1, \dots, x_l)$ satisfying

$$(m_1, \dots, m_l) \in A \Leftrightarrow \mathfrak{N} \models \varphi(\bar{m}_1, \dots, \bar{m}_l).$$

- Similarly, Π_i sets can be defined by Π_i formulas.
- A set that is both Σ_i and Π_i is called Δ_i .

Theorem 2.7 ((Strong) Representation Theorem for Δ_1 Sets)

Let T be a Σ_1 -complete theory. For any Δ_1 set C , there exists a Σ_1 formula $\varphi(x)$ which strongly represents C , i.e., $n \in C \Rightarrow T \vdash \varphi(\bar{n})$, $n \notin C \Rightarrow T \vdash \neg\varphi(\bar{n})$.

Proof. For a Δ_1 set C , there exist Σ_0 formulas $\theta_1(x, y), \theta_2(x, y)$ such that

$$n \in C \Leftrightarrow \mathfrak{N} \models \exists y \theta_1(\bar{n}, y), \quad n \notin C \Leftrightarrow \mathfrak{N} \models \exists y \theta_2(\bar{n}, y).$$

We may take $\varphi(x)$ as a Σ_1 formula $\exists y(\theta_1(\bar{n}, y) \wedge \forall z \leq y \neg \theta_2(\bar{n}, z))$. □

Theorem 2.8 (Representation Theorem for Δ_1 Function)

Let T be Σ_1 -complete. For any Δ_1 function $f(\vec{x})$, there exists a Σ_1 formula $\varphi(\vec{x}, y)$ which functionally represents $f(\vec{x}) = y$.

Proof. For simplicity, we assume that $l = 1$. Suppose $f(x) = y$ is represented by a Σ_1 formula $\varphi(x, y) \equiv \exists z \theta(x, y, z)$ with $\theta(x, y, z) \in \Sigma_0$. We define a Σ_0 formula $\psi(x, y, z)$ as

$$\theta(x, y, z) \wedge \forall y', z' \leq y + z (\theta(x, y', z') \rightarrow y + z \leq y' + z').$$

Then, $\exists z \psi(x, y, z)$ functionally represents $f(x) = y$. □

We investigate some properties of $I\Sigma_1$, e.g., the definability of primitive recursive functions.

Definition 3.1

For a formula $\varphi(x, y_1, \dots, y_k)$ of \mathcal{L}_{OR} , the following formula is called the **collection principle** or **bounding principle** of φ , denoted $(B\varphi)$:

$$\forall x < u \exists y_1 \cdots \exists y_k \varphi(x, y_1, \dots, y_k) \rightarrow \exists v \forall x < u \exists y_1 < v \cdots \exists y_k < v \varphi(x, y_1, \dots, y_k),$$

where $\varphi(x, y_1, \dots, y_k)$ may include undisplayed variables other than u, v . If the collection principle should be treated as a sentence, we consider its universal closure. For a class Γ of formulas, we let

$$B\Gamma = I\Sigma_0 \cup \{(B\varphi) : \varphi \in \Gamma\}.$$

For any n , the collection principle of a Σ_{n+1} formula

$$\varphi(x, y_1, \dots, y_k) (\equiv \exists z_1 \cdots \exists z_l \theta(x, y_1, \dots, y_k, z_1, \dots, z_l))$$

can be obtained from the collection principle of a Π_n formula $\theta(x, y_1, \dots, y_k, z_1, \dots, z_l)$ with $k + l$ variables. Therefore, $B\Sigma_{n+1} \Leftrightarrow B\Pi_n$.

Lemma 3.2

In $B\Sigma_n$ ($n \geq 1$), adding bounded quantifiers $\forall x < t$, $\exists x < t$ in front of a Σ_n formula produces a formula that is equivalent to a Σ_n formula. Similarly for a Π_n formula.

Proof. By meta-induction on n .

- The case $n = 1$.

Take any Σ_1 formula $\exists y_1 \cdots \exists y_k \varphi(x, y_1, \dots, y_k)$. By $B\Sigma_1$, we have

$$\forall x < t \exists y_1 \cdots \exists y_k \varphi(x, y_1, \dots, y_k) \rightarrow \exists v \forall x < t \exists y_1 < v \cdots \exists y_k < v \varphi(x, y_1, \dots, y_k),$$

and obviously the converse \leftarrow also holds. Thus, adding $\forall x < t$ in front of the Σ_1 formula is equivalent to a Σ_1 formula of the right-hand side. If $\exists x < t$ is added before a Σ_1 formula, it can be converted into a Σ_1 formula by shifting $\exists x < t$ to the end of the block of existential quantifiers of the formula.

Π_1 formulas can be treated similarly.

- For $n > 1$, by the same argument as above, we exchange the order of a bounded quantifier $\forall x < t$ and an existential quantifier in front of a Σ_n formula. Then, by induction hypothesis, we can transform the Π_{n-1} formula preceded by a bounded quantifier into an equivalent Π_{n-1} formula.
 Π_n formulas can be treated similarly.

Lemma 3.3

For any $n \geq 1$, $B\Sigma_n$ is a subsystem of $I\Sigma_n$.

Proof. We use meta-induction on n .

- Let $\exists z_1 \cdots \exists z_l \varphi(x, y_1, \dots, y_k, z_1, \dots, z_l)$ be Σ_n with $\varphi(x, y_1, \dots, y_k, z_1, \dots, z_l) \Pi_{n-1}$.
- Suppose $\forall x < u \exists y_1 \cdots \exists y_k \exists z_1 \cdots \exists z_l \varphi(x, y_1, \dots, y_k, z_1, \dots, z_l)$.¹
- By the induction hypothesis $B\Sigma_{n-1}$ (obvious for $n = 1$) and the above lemma, the following formula $\psi(w)$ is Σ_n .

$$\psi(w) := (\exists v \forall x < w \exists y_1 < v \cdots \exists y_k < v \exists z_1 < v \cdots \exists z_l < v \varphi) \vee u < w.$$

- Now, we want to prove $\forall w \psi(w)$ by induction.
- Clearly, $\psi(0)$ holds.
- Assume $\psi(w)$ and we will show $\psi(w + 1)$.

¹In the following, we may treat u as a constant.

- If $u < w + 1$, $\psi(w + 1)$ is obvious.
- So, assume $w < u$. By the first assumption, there exist $y'_1, \dots, y'_k, z'_1, \dots, z'_l$ such that $\varphi(w, y'_1, \dots, y'_k, z'_1, \dots, z'_l)$. By the induction hypothesis $\psi(w)$, there is v such that

$$\forall x < w \exists y_1 < v \cdots \exists y_k < v \exists z_1 < v \cdots \exists z_l < v \varphi.$$

- If we put

$$v' = \max\{v, y'_1 + 1, \dots, y'_k + 1, z'_1 + 1, \dots, z'_l + 1\},$$

then $\forall x < w + 1 \exists y_1 < v' \cdots \exists y_k < v' \exists z_1 < v' \cdots \exists z_l < v' \varphi$, which implies $\psi(w + 1)$.

- So by Σ_n induction, $\psi(w)$ holds for all w . In particular, if $w = u$,

$$\exists v \forall x < u \exists y_1 < v \cdots \exists y_k < v \exists z_1 < v \cdots \exists z_l < v \varphi,$$

which implies $\exists v \forall x < u \exists y_1 < v \cdots \exists y_k < v \exists z_1 \cdots \exists z_l \varphi$. □

By the above two lemmas, we have

Lemma 3.4

For any n , $I\Sigma_n$ and III_n are equivalent.

Proof.

- We show that III_n is provable in $I\Sigma_n$. The other case can be treated in a similar way.
- Let φ be a Π_n formula for induction. We consider the backward induction on the formula $\neg\varphi(c - x)$. More strictly, we use the induction on the following formula

$$\psi(x) \equiv \exists y \leq c(x + y = c \wedge \neg\varphi(y)) \vee c < x.$$

It is a Σ_n formula by the lemma in Page 10.

- Similarly, $I\Sigma_n$ is provable in III_n .

Homework # 3-3

(1) The following formula is called the **least number principle** for a formula φ and denoted as $L\varphi$,

$$\exists x\varphi(x) \rightarrow \exists x(\varphi(x) \wedge \forall y < x \neg \varphi(y)).$$

$L\Sigma_n$ stands for $\{L\varphi : \varphi \text{ is } \Sigma_n\}$. Then, show that $I\Sigma_n$ is equivalent to $L\Sigma_n$.

(2) For any n , show $B\Sigma_{n+1} \supset I\Sigma_n$.

It is also known that the relation $I\Sigma_{n+1} \supset B\Sigma_{n+1} \supset I\Sigma_n$ is strict.^{2 3}

²Petr Hájek and Pavel Pudlák. *Metamathematics of first-order arithmetic*. Springer, 1993

³Kaye R. *Models of Peano arithmetic*, Oxford University Press, 1991.

Next we discuss the definability of primitive recursive functions in $\text{I}\Sigma_1$. The following lemma is a basic tool for uniquely assigning natural numbers to finite sets and finite sequences in $\text{I}\Sigma_1$.

Lemma 3.5

In $\text{I}\Sigma_1$, for a Σ_1 formula $\varphi(x)$ and a Π_1 formula $\psi(x)$, we can prove

$$\forall x(\varphi(x) \leftrightarrow \psi(x)) \rightarrow \forall u \exists m, n > 0 \forall x < u (\varphi(x) \leftrightarrow m(x+1) + 1 \text{ is a divisor of } n).$$

Proof.

- First, fix u . The existence of a number m which divides all $i < u$ can be easily shown by Σ_1 induction.
- Then, for all $i < u$, $m(i+1) + 1$ are mutually prime. \therefore If $m(i+1) + 1$ and $m(j+1) + 1$ ($i < j < u$) are both multiples of a prime number d , $(m(j+1) + 1) - (m(i+1) + 1) = m(j-i)$ should also be a multiple of d . But d is never a divisor of m because it divides $m(i+1) + 1$. Also, d is not a divisor of $m(j-i)$ since $d \geq u > j-i$.

- Next, let $\varphi(x)$ be Σ_1 formula, and $\psi(x)$ be a Π_1 formula. Assume $\forall x(\varphi(x) \leftrightarrow \psi(x))$.
- Then, by Σ_1 induction on j , we prove the following.

$$\exists n \forall x < j \left[(\psi(x) \rightarrow m(x+1) + 1 \text{ is a divisor of } n) \wedge (m(x+1) + 1 \text{ is a divisor of } n \rightarrow \varphi(x)) \right] \vee u < j.$$

- It is obvious when $j = 0$.
- Let n_j be the minimum n that satisfies the above condition for j (See the least number principle, Problem 3).
- Now, if $\varphi(j)$, then $n_{j+1} = n_j \cdot (m(j+1) + 1)$, otherwise $n_{j+1} = n_j$.
- Note that for all $i < u$, $m(i+1) + 1$ are mutually prime, and n_j does not contain any factor of $m(j+1) + 1$ due to its minimality.
- Then, n_{j+1} satisfies the above condition for $j+1 \leq u$, which completes the induction step.
- Thus, the lemma holds as $j = u$. □

- In the above lemma, the triple (u, m, n) satisfying

$$\forall x < u \left(\varphi(x) \leftrightarrow m(x+1) + 1 \text{ is a divisor of } n \right)$$

is called a **u -piece code** of a Δ_1 set $\{x \mid \varphi(x)\}$ defined by a Σ_1 formula $\varphi(x)$ and a Π_1 formula $\psi(x)$.

- We will extend the above to n -dimensional sets. First, we code a pair of natural numbers (x, y) by a natural number $\langle x, y \rangle = \frac{(x+y)(x+y+1)}{2} + x$. Note that if $u = \langle u_1, u_2 \rangle$, then $\langle x, y \rangle < u$ for any $x < u_1, y < u_2$.
- Then, from the lemma in Page 15, for a Δ_1 formula $\varphi(x, y)$, there exist u, m, n s.t.

$$\forall x < u_1 \forall y < u_2 (\varphi(x, y) \leftrightarrow m(\langle x, y \rangle + 1) + 1 \text{ is a divisor of } n)$$

The triple $c = (u, m, n)$ is called a (u_1, u_2) -piece code of the Δ_1 set.

- In general, by coding an n -tuple (x_1, x_2, \dots, x_n) by a natural number $\langle \langle \dots \langle x_1, x_2 \rangle, \dots \rangle, x_n \rangle$, we can define a (u_1, u_2, \dots, u_n) -piece code of a Δ_1 n -dimensional set.

Theorem 3.6 (Definability of primitive recursive functions)

In $\text{I}\Sigma_1$, the graph of a primitive recursive function $f(x_1, \dots, x_l, y) = z$ can be represented by a Δ_1 formula $\varphi(x_1, \dots, x_l, y, z)$, and the following are provable

$$\forall x_1 \cdots \forall x_l \forall y \exists! z \varphi(x_1, \dots, x_l, y, z).$$

Proof.

- We will prove this by induction on the construction of primitive recursive functions. The essential step is the definition by primitive recursion.
- For simplicity, we omit parameter variables x_1, \dots, x_l , and consider the definition of a unary function f from a constant c and binary function h as follows:

$$f(0) = c, \quad f(y + 1) = h(y, f(y)).$$

- From the induction hypothesis, h can be expressed in both Σ_1 and Π_1 formulas.
- If f can be expressed by a Δ_1 formula φ , it will be easily derived in $\text{I}\Sigma_1$ that $\forall \vec{x} \forall y \exists! z \varphi(\vec{x}, y, z)$.

- First, let $\gamma(x, m, n)$ be a Σ_0 formula expressing “ $m(x + 1) + 1$ is a divisor of n ”, that is, $\exists d < n (m(x + 1) + 1) \cdot d = n$.
- We define a predicate $\delta(u, m, n)$ such that

$$\delta(\langle u_1, u_2 \rangle, m, n) \Leftrightarrow \forall y < u_1 \exists z < u_2 f(y) = z,$$

by the following Σ_0 formula: for any $u = \langle u_1, u_2 \rangle$,

$$\begin{aligned} \delta(u, m, n) \equiv & \forall y < u_1 \exists z < u_2 \gamma(\langle y, z \rangle, m, n) \wedge \forall z < u_2 (\gamma(\langle 0, z \rangle, m, n) \leftrightarrow z = c) \\ & \wedge \forall y < u_1 \neg \forall z < u_2 (\gamma(\langle y + 1, z \rangle, m, n) \leftrightarrow \exists z' < u_2 (z = h(y, z') \wedge \gamma(\langle y, z' \rangle, m, n))). \end{aligned}$$

- Then, by $\text{I}\Sigma_1$, we can show $\forall u_1 \exists u_2 \exists m \exists n \delta(\langle u_1, u_2 \rangle, m, n)$
- Therefore, we have

$$\begin{aligned} f(y) = z \Leftrightarrow & \exists u \exists m \exists n (u_1 = y + 1 \wedge \delta(u, m, n) \wedge \gamma(\langle y, z \rangle, m, n)) \\ \Leftrightarrow & \forall u \forall m \forall n (u_1 = y + 1 \wedge \delta(u, m, n) \rightarrow \gamma(\langle y, z \rangle, m, n)) \end{aligned}$$

- Thus, $f(y) = z$ is expressed by a Δ_1 formula.

- The above theorem shows that adding a symbol for a primitive recursive function and its definition to $\text{I}\Sigma_1$, we obtain a conservative extension.
- Furthermore, even if primitive recursive function symbols are involved, the classes of Σ_n and Π_n formulas ($n > 0$) do not essentially change. In other words, for a Σ_n formula containing primitive recursive function symbols, we can construct an equivalent Σ_n formula that includes no primitive recursive function symbols by replacing a primitive recursive function symbol with a Σ_1 formula or Π_1 formula that defines it.
- In the lemma in Page 15, we showed the existence of a u -piece code for a Δ_1 set. Also, a finite sequence of natural numbers $s = (s_0, \dots, s_{n-1})$ can be coded as a natural number c . Then, we identify s and c , and write c_i for s_i . Note that $(c, i) \mapsto c_i$ is primitive recursive.

Recap: Partial computable functions and CE sets

- If a **partial computable function** $f : \mathbb{N}^k \rightarrow \mathbb{N}$ is realized by a Turing machine \mathcal{M} with index e ,
 f is denoted by $\{e\}^k$ (or simply $\{e\}$). When e is not an index of TM, $\{e\}$ is regarded as a partial function with empty domain.
- The partial computable functions coincide with the **partial recursive functions** which are the smallest class that contains the constant 0, the successor function, projections, and closed under composition, primitive recursion and minimalization.
- A set $X \subset \mathbb{N}^n$ is said to be **computably enumerable** or **CE** if X is the domain of a partial computable function. X is said to be **computable** if both X and X^c are CE.
- A **halting program** $K = \{e : \{e\}(e) \downarrow\}$ is CE but not computable.
- R is CE iff there exists a primitive recursive relation S such that

$$R(x_1, \dots, x_n) \Leftrightarrow \exists y S(x_1, \dots, x_n, y).$$

Lemma 3.7

The CE sets are exactly the same as the Σ_1 sets. Hence, the computable (recursive) sets are exactly the same as the Δ_1 sets.

Proof.

- Any CE relation $R(\vec{x})$ can be expressed by $\exists y S(\vec{x}, y)$ for some primitive recursive relation S .
- By the definability theorem of prim. rec. functions, any primitive recursive relation S can be expressed by a Σ_1 formula, and so $\exists y S(\vec{x}, y)$ is still Σ_1 .
- Conversely, a Σ_1 formula is expressed in the form $\exists y \theta(\vec{x}, y)$ with $\theta(\vec{x}, y) \in \Sigma_0$. Since a Σ_0 formula is a primitive recursive, a Σ_1 formula is CE.

§4. The first incompleteness theorem

From now on, we assume that all theories are given in the language \mathcal{L}_{OR} and contain at least R , so Σ_1 -complete. We will prove a version of Gödel's first incompleteness theorem

Definition 4.1

Theory T is **1-consistent** if, for any Σ_1 sentence σ , $T \vdash \sigma \Rightarrow \mathfrak{N} \models \sigma$.

- Ordinary theories T of arithmetic such as Q and PA have the standard model \mathfrak{N} , so they are naturally 1-consistent, and indeed **ω -consistent** (i.e., for any formula $\varphi(x)$, if $T \vdash \varphi(\bar{n})$ for all $n \in \mathbb{N}$ then $T \not\vdash \exists x \neg \varphi(x)$.)
- 1-consistency is properly stronger than consistency.
E.g., $Q + \exists x(0 + x \neq x)$ is consistent but not 1-consistent.

Lemma 4.2 ((Weak) Representation Theorem for CE sets)

Suppose that a theory T is Σ_1 -complete and 1-consistent. Then, for any CE set C , there exists a Σ_1 formula $\varphi(x)$ such that for any n ,

$$n \in C \quad \Leftrightarrow \quad T \vdash \varphi(\bar{n}).$$

Theorem 4.3 (Gödel's first incompleteness theorem, a naïve version)

Let T be a Σ_1 -complete and 1-consistent Σ_1 theory. Then T is incomplete, that is, there is a sentence σ which T cannot prove or disprove.

Proof.

- We know K is CE but not co-CE. By the weak representation theorem for CE sets, there exists a formula $\varphi(x)$ such that

$$n \in K \Leftrightarrow T \vdash \varphi(\bar{n}).$$

- On the other hand, since $\mathbb{N} - K$ is not a CE, there exists some d such that

$$d \in \mathbb{N} - K \not\vdash T \vdash \neg\varphi(\bar{d}).$$

Thus, $(d \in K \text{ and } T \vdash \neg\varphi(\bar{d}))$ or $(d \notin K \text{ and } T \not\vdash \neg\varphi(\bar{d}))$.

- In the former case, since $d \in K$ implies $T \vdash \varphi(\bar{d})$, T is inconsistent, contradicting with the 1-consistency assumption.
- In the latter case, T is incomplete because $\varphi(\bar{d})$ cannot be proved or disproved.

We prepare some useful prim. rec. functions for coding things.

First, recall that if A, B are prim. rec. n -ary relations, also are $\neg A$, $A \wedge B$, $A \vee B$.

Lemma 4.4

For a primitive recursive function $h(\vec{x})$, $\mu y < h(\vec{x}) A(\vec{x}, y)$ is primitive recursive.

Example

Let $p(x) = "(x + 1)\text{-th prime number}"$, that is ,

$$p(0) = 2, p(1) = 3, p(2) = 5, \dots$$

Then, $p(x)$ is a primitive recursive function since it is defined as follows:

$$p(0) = 2, \quad p(x + 1) = \mu y < p(x)! + 2 (p(x) < y \wedge \text{prime}(y)).$$

- A finite sequence of natural numbers (x_0, \dots, x_{n-1}) can be represented by a single natural number x (called a **sequence number**) as follows,

$$x = p(0)^{x_0+1} \cdot p(1)^{x_1+1} \cdot \dots \cdot p(n-1)^{x_{n-1}+1}.$$

- Fixing n , such a mapping from \mathbb{N}^n to \mathbb{N} is a primitive recursive function.
- Conversely, the function $c(x, i)$ taking the i -th element x_i from a sequence number x is defined as

$$x_i = c(x, i) = \mu y < x (\neg \exists z < x (p(i)^{y+2} \cdot z = x)).$$

- The length of the sequence represented by x is

$$\text{leng}(x) = \mu i < x (\neg \exists z < x (p(i) \cdot z = x)).$$

- Furthermore, we define a primitive recursive relation $\text{Seq}(x)$ to denote that a natural number x is a sequence number as follows:

$$\text{Seq}(x) \Leftrightarrow \forall i < x \forall z < x (p(i) \cdot z = x \rightarrow i \leq \text{leng}(x)).$$

Definition 4.5

Let Ω be a finite (or countably infinite) set of symbols, and an injection $\phi : \Omega \rightarrow \mathbb{N}$. For a string $s = a_0 \cdots a_{n-1}$, the following natural number $\psi(s)$ is called the **Gödel number** of s , denoted by $\ulcorner s \urcorner$.

$$\psi(s) = p(0)^{\phi(a_0)+1} \cdot p(1)^{\phi(a_1)+1} \cdot \dots \cdot p(n-1)^{\phi(a_{n-1})+1}.$$

The mapping $\ulcorner \ \urcorner$ is an injection from the set of all symbols Ω^* to \mathbb{N} .

Example

Let $\Omega = \{0, 1, +, (,)\}$, $\phi(0) = 0$, $\phi(1) = 1$, $\phi(+)$ = 3, $\phi(($) = 5 and $\phi($) = 6.

Then,

$$\ulcorner (1 + 0) + 1 \urcorner = 2^6 \cdot 3^2 \cdot 5^4 \cdot 7^1 \cdot 11^7 \cdot 13^4 \cdot 17^2$$

Problem 5

Show that $\text{Term}(x)$ expressing “ x is the Gödel number of a term” is primitive recursive.

Definition 4.6

A theory T is Σ_i (Π_i/Δ_i /**primitive recursive**, etc.) if the set of Gödel numbers of its axioms $\{\ulcorner \sigma \urcorner : \sigma \in T\}$ is Σ_i (Π_i/Δ_i /primitive recursive, etc.).

- Ordinary theories in mathematics are finite or at most primitive recursive.
- The theories of arithmetic introduced so far (PA, $I\Sigma_1$, etc.) are all primitive recursive.
- To derive the incompleteness theorem, we need to assume that a theory is CE.
- Without this condition, for example, if we take all true arithmetic formulas as axioms, we would have a complete theory, but it would not be a formal system.
- By the following theorem, a Σ_1 set of axioms can be always be replaced by a primitive recursive set.

Theorem 4.7 (Craig's lemma)

For any CE theory T , there exists a primitive recursive theory T' that proves the same theorems.

Proof. Let T be a theory defined by a Σ_1 formula $\varphi(x) \equiv \exists y\theta(x, y)$ (θ is Σ_0). That is, $\sigma \in T \Leftrightarrow \mathfrak{N} \models \varphi(\ulcorner\sigma\urcorner)$, where $\ulcorner\sigma\urcorner$ is the Gödel number of a sentence σ . Then, we define a primitive recursive theory T' as follows:

$$T' = \{\overbrace{\sigma \wedge \sigma \wedge \cdots \wedge \sigma}^{n+1 \text{ copies}} : \theta(\ulcorner\sigma\urcorner, \bar{n})\}.$$

Then, T and T' are equivalent, since $\vdash \sigma \leftrightarrow \sigma \wedge \sigma \wedge \cdots \wedge \sigma$. Thus T' is primitive recursive. □

Because Gödel numbers and their decodings are heavily used in T' , T' cannot be easily expressed in Σ_0 .

Based on Craig's lemma, a Σ_1 theory is primitive recursively axiomatizable. Then, "a finite sequence (or finite tree) P of formulas is a proof in T " can be defined in a primitive recursive way (with T as a parameter).

Definition 4.8

- Let T be a Σ_1 theory and T' its p.r. counterpart. A **proof** in T' is a finite sequence of formulas where each formula is either a logical axiom, an equality axiom, or an axiom of T' , or obtained by applying MP or quantification rules from formulas appearing before. The formula that appears at the end of a proof is a **theorem** of T .
- Now, we define the primitive recursive predicate Proof_T as follows.

$$\text{Proof}_T(\ulcorner P \urcorner, \ulcorner \sigma \urcorner) \Leftrightarrow P \text{ is a proof of formula } \sigma \text{ in } T'.$$

- By Proof_T , we also denote a Δ_1 formula expressing the above Proof_T in IS_1 . A Σ_1 formula Bew_T is defined as

$$\text{Bew}_T(x) \equiv \exists y \text{Proof}_T(y, x).$$

The formula $\text{Bew}_T(x)$ expresses that " x is the Gödel number of a theorem of T ". "Bew" stands for the German beweisbar (provable).

Recap: Representation Theorems

Theorem 2.7 ((Strong) Representation Theorem for Computable Sets, modified)

Assume a theory T is Σ_1 -complete. For any computable set C , there exists a Σ_1 formula $\varphi(x)$ such that

$$n \in C \Rightarrow T \vdash \varphi(\bar{n}), \quad n \notin C \Rightarrow T \vdash \neg\varphi(\bar{n}).$$

Theorem 2.8 (Representation Theorem for Computable Function)

Let T be Σ_1 -complete. For any computable function $f(\vec{x})$, there exists a Σ_1 formula $\varphi(\vec{x}, y)$ which represents $f(\vec{x}) = y$ and satisfies, for all natural numbers m_1, \dots, m_l ,

$$T \vdash \forall y \forall y' (\varphi(\bar{m}_1, \dots, \bar{m}_l, y) \wedge \varphi(\bar{m}_1, \dots, \bar{m}_l, y') \rightarrow y = y').$$

Lemma 4.9 (Diagonalization lemma)

Let T be Σ_1 -complete. For any formula $\psi(x)$ in which x is the unique free variable, there exists a sentence σ such that $T \vdash “\sigma \leftrightarrow \psi(\overline{\overline{\sigma}})”$.

Proof.

- A formula with only x as a free variable is computably enumerated as $\varphi_0(x), \varphi_1(x), \dots$, and then $f(n) = \overline{\overline{\varphi_n(\overline{n})}}$ is also a computable function. By the functional representation theorem, there exists a Σ_1 formula χ such that

$$f(m) = n \Rightarrow T \vdash \chi(\overline{m}, \overline{n}) \wedge \forall y \neq \overline{n} \neg \chi(\overline{m}, y).$$

- The formula $\exists y(\chi(x, y) \wedge \psi(y))$ must be listed as $\varphi_k(x)$ for some k . Now, let σ be $\varphi_k(\overline{k})$. Since $f(k) = \overline{\overline{\sigma}}$, $T \vdash \chi(\overline{k}, \overline{\overline{\sigma}})$. Thus, in T , $\psi(\overline{\overline{\sigma}}) \rightarrow \exists y(\chi(\overline{k}, y) \wedge \psi(y))$ ($\equiv \varphi_k(\overline{k}) \equiv \sigma$).
- On the other hand, since $T \vdash \forall y \neq \overline{\overline{\sigma}} \neg \chi(\overline{k}, y)$, in T ,

$$\neg \psi(\overline{\overline{\sigma}}) \rightarrow \forall y(\chi(\overline{k}, y) \rightarrow \neg \psi(y)) \rightarrow \neg \exists y(\chi(\overline{k}, y) \wedge \psi(y)) \quad (\equiv \neg \sigma).$$

- Therefore, $T \vdash \sigma \leftrightarrow \psi(\overline{\overline{\sigma}})$, that is, σ is a fixed point of ψ . □

Theorem 4.10 (Gödel's first incompleteness theorem, a formal version)

Let T be a Σ_1 -complete and 1-consistent Σ_1 theory. Then T is incomplete, that is, there is a sentence σ such that $T \not\vdash \sigma$ and $T \not\vdash \neg\sigma$.

Proof. By the diagonalization lemma, there exists a fixed point σ of $\neg\text{Bew}_T(x)$. In other words, $T \vdash \sigma \leftrightarrow \neg\text{Bew}_T(\overline{\ulcorner\sigma\urcorner})$. We show that σ is such a sentence that T cannot prove or disprove as follows.

- Suppose $T \vdash \sigma$. Then $\text{Bew}_T(\overline{\ulcorner\sigma\urcorner})$ holds, that is, $\mathfrak{N} \models \text{Bew}_T(\overline{\ulcorner\sigma\urcorner})$. Therefore, by Σ_1 completeness, $T \vdash \text{Bew}_T(\overline{\ulcorner\sigma\urcorner})$. Since σ is a fixed point of $\neg\text{Bew}_T(x)$, we have $T \vdash \neg\sigma$ which implies the inconsistency of T , a contradiction.
- On the other hand, suppose $T \vdash \neg\sigma$. Since σ is a fixed point, $T \vdash \text{Bew}_T(\overline{\ulcorner\sigma\urcorner})$. By 1-consistency of T , $\mathfrak{N} \models \text{Bew}_T(\overline{\ulcorner\sigma\urcorner})$, that is, $T \vdash \sigma$, which also implies the inconsistency of T . □

Thank you for your attention!