Topics in Applied Math: Logic and Foundations of Mathematics

Part 3. Basic Model theory

Kazuyuki Tanaka

BIMSA

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清華大學 求真书院
Qiuzhen College, Tsinghua University

Logic and Foundations

- **Part 1. Equational theory**
- **Part 2. First order theory**
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Part 3. Schedule

- Oct. 15, (1) Various forms of axiomatic systems
- Oct. 17, (2) Reduced products and ultra products
- later, non-standard analysis

§1. Basic definitions

- The **theory** of a structure $\mathfrak{A}$, denoted $\mathrm{Th}(\mathfrak{A})$, is the set of sentences true in $\mathfrak{A}$.
- The **elementary diagram** of $\mathfrak{A}$ is $\mathrm{Th}(\mathfrak{A}_\mathrm{A})$.
- $\mathrm{Diag}(\mathfrak{A})$ = the set of atomic sentences and negations of atomic sentences in $\mathrm{Th}(\mathfrak{A}_\mathrm{A})$, is called the **basic diagram**.
- $\mathcal{L}$-structures $\mathfrak{A}$ and $\mathfrak{B}$ are **elementary equivalent**, denoted $\mathfrak{A} \equiv \mathfrak{B}$, if the same $\mathcal{L}$-sentences hold in both structures, that is, $\mathrm{Th}(\mathfrak{A}) = \mathrm{Th}(\mathfrak{B})$
- A structure $\mathfrak{A}$ is an **elementary substructure** of a structure $\mathfrak{B}$, denoted $\mathfrak{A} \prec \mathfrak{B}$, if $\mathfrak{A}$ is a substructure of $\mathfrak{B}$ and the same $\mathcal{L}_A$-sentences hold in both structures, i.e, $\mathrm{Th}(\mathfrak{A}_A) = \mathrm{Th}(\mathfrak{B}_A)$.
- Note that the notion of elementary substructure is stronger than that of elementary equivalence:

$$\mathfrak{A} \cong \mathfrak{B} \Rightarrow \mathfrak{A} \prec \mathfrak{B} \Rightarrow \mathfrak{A} \equiv \mathfrak{B}$$

Recap

Theorem 1.7 (Tarski-Vaught's criterion)

$\mathfrak{A} \prec \mathfrak{B}$ iff $\mathfrak{A} \subseteq \mathfrak{B}$ and for any formula $\varphi(x, y_1, \ldots, y_m)$ and any $a_1, \ldots, a_m \in A$,

$$\mathfrak{B}_A \models \exists x \varphi(x, a_1, \ldots, a_m) \Rightarrow \text{there exists an } a \in |\mathfrak{A}| \text{ s.t. } \mathfrak{B}_A \models \varphi(a, a_1, \ldots, a_m).$$

Definition 1.8

A chain of structures $\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \cdots \subseteq \mathfrak{A}_i \subseteq \cdots (i < \omega)$ is called an **elementary chain** if

$$\mathfrak{A}_0 \prec \mathfrak{A}_1 \prec \cdots \prec \mathfrak{A}_i \prec \cdots \quad (i < \omega).$$

And the structure $\mathfrak{A} = \bigcup_{i<\omega} \mathfrak{A}_i$ is called the **union** of the elementary chain.

Theorem 1.9 (Elementary chain theorem)

Let $\mathfrak{A}_0 \prec \mathfrak{A}_1 \prec \cdots$ be an elementary chain. Let $\mathfrak{A}$ be the union of the elementary chain. Then for each i, $\mathfrak{A}_i \prec \mathfrak{A}$.

§2. ∀-theory and ∀∃-theory

Definition 2.5

For an open formula (a formula without quantifiers) φ,
$\forall x_1 \cdots \forall x_m \varphi$ is called a $\forall$ **formula** (or **universal**, Π_1), and
$\forall x_1 \cdots \forall x_n \exists y_1 \cdots \exists y_m \varphi$ is called a $\forall\exists$ **formula** (or **universal-existential**, Π_2).

A set of $\forall$ sentences is called a **∀-theory** or a **universal theory**, and a set of $\forall\exists$ sentences is called a **∀∃-theory** or an **inductive theory**.

Let T be a theory of a language $\mathcal{L}$. We denote the class of all models of T by $\mathrm{Mod(T)}$, i.e.,

$$\mathrm{Mod(T)} = \{\mathfrak{A} : \mathfrak{A} \models \mathrm{T}\}$$

Theorem 2.6 (Łoś-Tarski)

The following two conditions are equivalent.

1. $\mathrm{Mod(T)}$ is closed under substructures.
2. There exists an $\forall$-theory T' such that $\mathrm{Mod(T)} = \mathrm{Mod(T')}$.

Theorem 2.7 (Chang-Łoś-Suszko)

The followings are equivalent.

(1) $\mathrm{Mod(T)}$ is closed under the union of chains. That is, if $\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \cdots$ is a chain of models of T, the union is also a model of T.

(2) There exists a $\forall\exists$-theory T' such that $\mathrm{Mod(T')} = \mathrm{Mod(T)}$.

Proof.

To show (2) $\Rightarrow$ (1), Let $\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \cdots$ be a chain of models of $\forall\exists$ theory T'.
We want to show $\mathfrak{A} = \bigcup_{i\in\mathbb{N}} \mathfrak{A}_i$ is also a model of T'.

- Let $\forall x_1 \cdots \forall x_n \exists y_1 \cdots \exists y_m \varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ be a $\forall\exists$-sentence in T'.
- Take any $a_1, \ldots, a_n \in A$. Then there exists some k such that $\{a_1, \ldots, a_n\} \subseteq A_k$.
- Since $\mathfrak{A}_k$ is a model of T', $\mathfrak{A}_{kA_k} \models \exists y_1 \cdots \exists y_m \varphi(a_1, \ldots, a_n, y_1, \ldots, y_m)$, and so there exist $b_1, \ldots, b_m \in A_k$ such that $\mathfrak{A}_{kA_k} \models \varphi(a_1, \ldots, a_n, b_1, \ldots, b_m)$.
- Since φ is open and $\mathfrak{A}_k \subseteq \mathfrak{A}$, we have $\mathfrak{A}_A \models \varphi(a_1, \ldots, a_n, b_1, \ldots, b_m)$.
- Thus $\mathfrak{A}_A \models \exists y_1 \cdots \exists y_m \varphi(a_1, \ldots, a_n, y_1, \ldots, y_m)$. Since $a_1, \ldots, a_n \in A$ are arbitrary,

$$\mathfrak{A} \models \forall x_1 \cdots \forall x_n \exists y_1 \cdots \exists y_m \varphi(x_1, \ldots, x_n, y_1, \ldots, y_m).$$

To show (1) $\Rightarrow$ (2), Let $T' = \{\sigma : \sigma$ is a $\forall\exists$-sentences, where $T \vdash \sigma\}$.

- Since $\mathrm{Mod(T)} \subseteq \mathrm{Mod(T')}$ is obvious, it suffices to show that $\mathrm{Mod(T')} \subseteq \mathrm{Mod(T)}$.
- Let $\mathfrak{A} \in \mathrm{Mod(T')}$ and $D_\forall$ be the set of all $\forall$-sentence contained in $\mathrm{Th}(\mathfrak{A}_\mathrm{A})$.
  - Using the compactness theorem in the same way as the proof of Łoś-Tarski theorem, $D_\forall \cup T$ has a model $\mathfrak{B}_A$.
  - Then $\mathfrak{B}$ as a reduct of $\mathfrak{B}_A$ is also a model of T, and so we have $\mathfrak{A} \subseteq \mathfrak{B}$
- Next let $D = \mathrm{Diag}(\mathfrak{B})$. We want to show $D \cup \mathrm{Th}(\mathfrak{A}_\mathrm{A})$ has a model.
  - Conversely, assume there is a conjunction φ of sentences from D s.t. $\mathrm{Th}(\mathfrak{A}_A) \vdash \neg\varphi$.
  - Assuming $b_1, \ldots, b_n$ are the constants appearing φ belonging to $B - A$, we express φ by $\varphi(b_1, \ldots, b_n)$, that is, $\varphi(x_1, \ldots, x_n)$ is a formula in language $\mathcal{L}_A$.
  - Since $\mathrm{Th}(\mathfrak{A}_A) \vdash \neg\varphi(b_1, \ldots, b_n)$, we have

$$\mathrm{Th}(\mathfrak{A}_A) \vdash \forall x_1 \cdots \forall x_n \neg\varphi(x_1, \ldots, x_n) \;\Rightarrow\; \forall x_1 \cdots \forall x_n \neg\varphi(x_1, \ldots, x_n) \in D_\forall$$

$$\Rightarrow\; \mathfrak{B}_A \models \forall x_1 \cdots \forall x_n \neg\varphi(x_1, \ldots, x_n) \;\Rightarrow\; \mathfrak{B}_B \models \neg\varphi(b_1, \ldots, b_n) \;\Rightarrow\; \text{Contradiction.}$$

- Now let $\mathfrak{A}'_A$ be the model of $D \cup \mathrm{Th}(\mathfrak{A}_\mathrm{A})$. Then $\mathfrak{B} \subseteq \mathfrak{A}'$ and $\mathfrak{A} \prec \mathfrak{A}'$.

- To summarize, for $\mathfrak{A} \in \mathrm{Mod(T')}$, there is a model $\mathfrak{B}(\supseteq \mathfrak{A})$ of T, and also a $\mathfrak{A}' \supseteq \mathfrak{B}$ such that $\mathfrak{A} \prec \mathfrak{A}'$.
- Since $\mathfrak{A}' \in \mathrm{Mod(T')}$, we can use the same argument to find a model $\mathfrak{B}'$ of T such that $\mathfrak{B}' \supseteq \mathfrak{A}'$, and there is a $\mathfrak{A}'' \supseteq \mathfrak{B}'$ such that $\mathfrak{A}' \prec \mathfrak{A}''$.
- By repeating this process, we have an elementary chain of models of T'

$$\mathfrak{A} \prec \mathfrak{A}' \prec \mathfrak{A}'' \prec \cdots$$

and a chain of models of T

$$\mathfrak{B} \subseteq \mathfrak{B}' \subseteq \mathfrak{B}'' \subseteq \cdots .$$

Since $\mathfrak{A}^{(n)} \subseteq \mathfrak{B}^{(n)} \subseteq \mathfrak{A}^{(n+1)}$, the two chains have the same union, denoted $\mathfrak{A}_\infty$.

- On one hand, by the elementary chain theorem, $\mathfrak{A} \prec \mathfrak{A}_\infty$.
- On the other hand, $\mathfrak{A}_\infty \models T$ by condition (1) of the theorem.

- Therefore, $\mathfrak{A}$ is a model of T, as desired. □

Definition 2.8

A theory T is said to be **model complete** if for any model $\mathfrak{A}$, $\mathfrak{B}$ of T,

$$\mathfrak{A} \subseteq \mathfrak{B} \Rightarrow \mathfrak{A} \prec \mathfrak{B}.$$

Example. The theory of algebraically closed fields is model complete.

Lemma 2.9

A model-complete theory is a $\forall\exists$-theory.

Proof. In a model-complete theory T, a chain of models is an elementary chain, so by the elementary chain theorem, the union is also a model of T. By the Chang-Łoś-Suszko theorem, this theory is a $\forall\exists$ theory.

Problem 7. **Homework # 3**

In a model-complete theory, show that for every formula, there exists an equivalent $\forall$ formula. (Hint. See the proof of (1)⇒(2) in the Łoś-Tarski theorem.)

§3. Horn formula and reduced product

We fix a language $\mathcal{L}$.

Definition 3.1

- $\theta_0 \vee \neg\theta_1 \vee \cdots \vee \neg\theta_n$ or $\neg\theta_1 \vee \cdots \vee \neg\theta_n$ is called a **basic Horn formula**, where θ_i $(i \leq n)$ are atomic formulas.
- A formula constructed from basic Horn formulas by using only $\wedge$, $\forall$, and $\exists$ is called **Horn formula**.
- The set of Horn sentences is called **Horn theory**.

The basic Horn formula can be expressed as follows, which is easier to use in applications:

$$\theta_1 \wedge \cdots \wedge \theta_n \to \theta_0$$

or

$$\theta_1 \wedge \cdots \wedge \theta_n \to \bot$$

Example 3.

The theory of regular rings, which adds the axiom $\forall x \forall y(xyx = x)$ to ring theory, is a Horn theory.

Theory of integral domain (commutative ring theory + $\forall x \forall y(x \bullet y = 0 \rightarrow x = 0 \vee y = 0))$ and field theory (commutative ring theory + $\forall x \exists y(x \neq 0 \rightarrow xy = 1))$ is not Horn theory.

The model of Horn's theory is closed under the so-called "reduced products", which is a generalization of direct product.

Before introducing reduced products, we starts with some definitions.

Definition 3.2

Let I be a non-empty set. $\mathcal{F} \subseteq \mathcal{P}(I)$ is called a **filter** on I if the following are satisfied.

(1) $\varnothing \notin \mathcal{F}$, $I \in \mathcal{F}$.

(2) $X \in \mathcal{F},\ X \subseteq Y \subseteq I \Rightarrow Y \in \mathcal{F}$.

(3) $X, Y \in \mathcal{F} \to X \cap Y \in \mathcal{F}$.

Problem 8

Let I be an infinite set. Show the following.

1. The class of all finite subsets of I is not a filter.
2. The class of all infinite subsets of I is not a filter.
3. The class of subsets X of I such that $I - X$ is finite is a filter, which is called a **Fréchet filter**.
4. For each $i \in I$, $\{X \subseteq I : i \in X\}$ is a filter, which is called a **principal filter**.

Lemma 3.3

If $S \subset \mathcal{P}(I)$ has the **finite intersection property**, i.e., for any finite $\{J_1, \dots J_n\} \subset S$,

$$J_1 \cap \dots \cap J_n \neq \varnothing,$$

then there exists a filter $\mathcal{F}$ including S.

Proof. Put $\mathcal{F} = \{X \subseteq I : J_1 \cap \dots \cap J_n \subset X \text{ for some } \{J_1, \dots, J_n\} \subset S\}$. □

Definition 3.4

Let $\mathfrak{A}_i = (A_i, \mathrm{f}^{\mathfrak{A}_i}, \dots, \mathrm{R}^{\mathfrak{A}_i}, \dots)$ $(i \in I)$ be an $\mathcal{L}$-structure. Let $\mathcal{F}$ be a filter for I. Then, we define the binary relation $\approx_{\mathcal{F}}$ on $\prod A_i$ as follows

$$a \approx_{\mathcal{F}} b \Leftrightarrow \{i \in I : a(i) = b(i)\} \in \mathcal{F}.$$

Lemma 3.5

$\approx_{\mathcal{F}}$ is an equivalence relation.

Proof

- The laws of reflection and symmetricity are clear from the definition.
- To show the transitive law, we assume $a \approx_{\mathcal{F}} b$, $b \approx_{\mathcal{F}} c$.
- By definition,

$$\{i \in I : a(i) = b(i)\} \in \mathcal{F} \text{ and } \{i \in I : b(i) = c(i)\} \in \mathcal{F}.$$

- On the other hand,

$$\{i \in I : a(i) = c(i)\} \supseteq \{i \in I : a(i) = b(i)\} \cap \{i \in I : b(i) = c(i)\}$$

- By conditions (2) and (3) of Definition 3.2, we have

$$\{i \in I : a(i) = c(i)\} \in \mathcal{F}.$$

Therefore, $a \approx_{\mathcal{F}} c$. □

For $a_1, \ldots, a_n \in \prod A_i$, we set $\|\varphi(a_1, \ldots, a_n)\| := \{i \in I : \mathfrak{A}_i \models \varphi(a_1(i), \ldots, a_n(i))\}$.
Note that we write $\mathfrak{A}$ for $\mathfrak{A}_A$ if it is clear from the context.

Lemma 3.6

If $a_1 \approx_{\mathcal{F}} b_1, \ldots, a_n \approx_{\mathcal{F}} b_n$, we have

$$\|\mathtt{f}(a_1, \ldots, a_n) = \mathtt{f}(b_1, \ldots, b_n)\| \in \mathcal{F},$$

$$\|\mathrm{R}(a_1, \ldots, a_n)\| \in \mathcal{F} \Leftrightarrow \|\mathrm{R}(b_1, \ldots, b_n)\| \in \mathcal{F}.$$

Proof.
This can be derived from the following with conditions (2) and (3) of Definition 3.2.

$$\bigcap_{k \leq n} \{i \in I : a_k(i) = b_k(i)\} \subseteq \|\mathtt{f}(a_1, \ldots, a_n) = \mathtt{f}(b_1, \ldots, b_n)\|,$$

$$\bigcap_{k \leq n} \{i \in I : a_k(i) = b_k(i)\} \cap \|\mathrm{R}(a_1, \ldots, a_n)\| \subseteq \ |\mathrm{R}(b_1, \ldots, b_n)\|.$$

□

Therefore, $\approx_{\mathcal{F}}$ is a congruence relation on $\prod A_i$.

- We can define the quotient structure in the same way as for the algebraic structure.
- That is, its domain is the set of equivalence classes denoted $\prod A_i/{\approx_{\mathcal{F}}}$ or $\prod A_i/\mathcal{F}$, and the value of a function f and the truth value of a relation R is uniquely determined on the equivalence classes regardless of choice of representative elements.

Definition 3.7

Let $\mathfrak{A}_i = (A_i, \mathtt{f}^{\mathfrak{A}_i}, \dots, \mathrm{R}^{\mathfrak{A}_i}, \dots)$ $(i \in I)$ be an $\mathcal{L}$-structure. Let $\mathcal{F}$ be a filter for I. Then, $\mathcal{L}$-structure obtained by the Cartesian product $\prod A_i$ dividing by the congruence relation $\approx_{\mathcal{F}}$ is defined by

$$\left(\prod A_i/\mathcal{F}, \mathtt{f}^{\prod \mathfrak{A}_i/\mathcal{F}}, \dots, \mathrm{R}^{\prod \mathfrak{A}_i/\mathcal{F}}, \dots\right)$$

is called the **reduced product** of $\mathfrak{A}_i$'s over $\approx_{\mathcal{F}}$, and is denoted by $\prod \mathfrak{A}_i/\mathcal{F}$.

- For a non-empty set I, $\mathcal{F} = \{I\}$ is a filter and $\prod \mathfrak{A}_i/\mathcal{F} \cong \prod \mathfrak{A}_i$.
  In other words, the direct product is also one kind of the reduced products.
- For a principal filter $\mathcal{F} = \{X \subseteq I : k \in X\}$, $\prod \mathfrak{A}_i/\mathcal{F} \cong \mathfrak{A}_k$.

Lemma 3.8

If φ is a formula obtained from atomic formulas with $\wedge$ and $\exists$, then for $a_1, \ldots, a_n \in \prod A_i$,

$$\prod \mathfrak{A}_i/\mathcal{F} \models \varphi([a_1], \ldots, [a_n]) \Leftrightarrow \|\varphi(a_1, \ldots, a_n)\| \in \mathcal{F}.$$

Proof. By induction on the construction of formulas.

- If φ is an atomic formula, it is clear by the definition.
- If $\varphi{=}\psi_1 \wedge \psi_2$, it follows from the induction hypo. and the closedness of filter under $\cap$.
- Let $\varphi = \exists x \psi(x)$. For simplicity, we do not display parameters $a_1, \ldots, a_n$ in φ.

$$\begin{aligned}\prod \mathfrak{A}_i/\mathcal{F} \models \exists x \psi(x) &\Leftrightarrow \text{ for some } a \in \prod \mathfrak{A}_i\ ,\ \prod \mathfrak{A}_i/\mathcal{F} \models \psi([a]) \\ &\Leftrightarrow \text{ for some } a \in \prod \mathfrak{A}_i\ ,\ \|\psi(a)\| \in \mathcal{F} \quad \text{(induction hypothesis)} \\ &\Rightarrow \|\exists x \psi(x)\| \in \mathcal{F} \quad (\because \|\psi(a)\| \subseteq \|\exists x \psi(x)\|).\end{aligned}$$

- Conversely, let $\|\exists x \psi(x)\| \in \mathcal{F}$. By the axiom of choice, we take $a \in \prod A_i$ such that for each $i \in \|\exists x \psi(x)\|$, $\mathfrak{A}_i \models \psi(a(i))$. Then, $\|\psi(a)\| \in \mathcal{F}$. By the induction hypothesis, $\prod \mathfrak{A}_i/\mathcal{F} \models \psi([a])$. Therefore, $\prod \mathfrak{A}_i/\mathcal{F} \models \exists x \psi(x)$. □

Lemma 3.9

Let $\varphi(x_1, \ldots, x_n)$ be a basic Horn formula, then for $a_1, \ldots, a_n \in \prod A_i$, we have

$$\|\varphi(a_1, \ldots, a_n)\| \in \mathcal{F} \Rightarrow \prod \mathfrak{A}_i/\mathcal{F} \models \varphi([a_1], \ldots, [a_n]).$$

Proof.

- For simplicity, we do not display parameters $a_1, \ldots, a_n \in \prod A_i$ in a formula.
- Let φ be a basic horn sentence $(\theta_0 \vee) \neg\theta_1 \vee \cdots \vee \neg\theta_n$, where θ_i $(i < n)$ are atomic sentences. We show a contradiction by assuming $\underline{①\|\varphi\| \in \mathcal{F}}$ and $\underline{②\prod \mathfrak{A}_i/\mathcal{F} \not\models \varphi}$.
- Suppose φ does not contain θ_0. Then, since $\prod \mathfrak{A}_i/\mathcal{F} \models \theta_1 \wedge \cdots \wedge \theta_n$ by ②, we have $\|\theta_1 \wedge \cdots \wedge \theta_n\| \in \mathcal{F}$ by Lemma 3.8, and so $\varnothing = \|\varphi\| \cap \|\theta_1 \wedge \cdots \wedge \theta_n\| \in \mathcal{F}$, which violates the condition of a filter.
- If φ contains θ_0, we have $\prod \mathfrak{A}_i/\mathcal{F} \models \neg\theta_0 \wedge \theta_1 \wedge \theta_1 \wedge \cdots \wedge \theta_n$ by ②, then also $\prod \mathfrak{A}_i/\mathcal{F} \models \theta_1 \wedge \theta_1 \wedge \cdots \wedge \theta_n$, and hence by Lemma 3.8 $\|\theta_1 \wedge \cdots \wedge \theta_n\| \in \mathcal{F}$. So $\|\theta_0\| \supset \|\varphi\| \cap \|\theta_1 \wedge \cdots \wedge \theta_n\| \in \mathcal{F}$, which implies $\|\theta_0\| \in \mathcal{F}$ by the property of a field. Again by Lemma 3.8, $\prod \mathfrak{A}_i/\mathcal{F} \models \theta_0$, which contradicts with ②. □

Lemma 3.10

Let $\varphi(x_1, \dots, x_n)$ be a Horn formula, then for $a_1, \dots, a_n \in \prod A_i$,

$$\|\varphi(a_1, \dots, a_n)\| \in \mathcal{F} \Rightarrow \prod \mathfrak{A}_i/\mathcal{F} \models \varphi([a_1], \dots, [a_n]).$$

Proof. By induction on the construction of a Horn formula with $\wedge$, $\forall$, and $\exists$.
For the basic Horn formula, it follows from Lemma 3.9. Formulas $\varphi \wedge \psi$ and $\exists x \varphi(x)$ are treated in Lemma 3.8 in Page 17.
For a formula $\forall x \varphi(x)$,

$$\begin{aligned} \|\forall x \varphi(x)\| \in \mathcal{F} &\Rightarrow \text{for all } a \in \prod A_i,\ \|\varphi(a)\| \in \mathcal{F} \\ &\Rightarrow \text{for all } a \in \prod A_i,\ \prod \mathfrak{A}_i/\mathcal{F} \models \varphi([a]) \\ &\Leftrightarrow \prod \mathfrak{A}_i/\mathcal{F} \models \forall x \varphi(x) \quad \square \end{aligned}$$

The above lemma shows that the Horn formula preserves reduced products, that is, a reduced product of models of a Horn formula becomes a model of the formula again. Therefore, the class of models of a Horn theory is closed under reduced products.

- The class of models of a Horn theory is closed under reduced products (especially under direct products). Then, the converse is also true in the following sense.

Theorem 3.11 (Keisler-Galvin)

The following are equivalent:

(1) $\mathrm{Mod}(\mathrm{T})$ is closed under reduced products.

(2) There exists a Horn theory T' such that $\mathrm{Mod}(\mathrm{T}) = \mathrm{Mod}(\mathrm{T}')$.

A proof can be found in Chang-Keisler's classic textbook *Model Theory*.

(Example) The product of regular rings is a regular ring.

HW # 4

Show that the class of Boolean algebras with atoms (non-atomless) is closed under direct products but not under reduced products.

- The theory of Boolean algebra is a ∀-theory. "Boolean algebra has an atom a" is expressed by the following ∃∀-sentence.

$$\exists a \forall x (a \neq 0 \land (a \bullet x = x \to x = a \lor x = 0)).$$

- A sentence with several $\forall$ in front of a basic Horn formula is called a **$\forall$-Horn sentence** (or simply called a Horn sentence in some literature). A collection of such sentences is called a **$\forall$-Horn theory** (or simply a Horn theory).
- A $\forall$-Horn theory is a nice extension of equational theory. The following theorem is a counter part of Birkhoff's equational class theorem. It can be proven similarly, and we leave the details to the reader.

Theorem 3.12

Let $\mathcal{K}$ be a class of $\mathcal{L}$-structures, then the following are equivalent:

(1) $\mathcal{K}$ is closed under direct products, substructures, and isomorphic images.

(2) $\mathcal{K}$ is closed under reduced products, substructures, and isomorphic images.

(3) There exists a $\forall$-Horn theory T such that $\mathrm{Mod(T)} = \mathcal{K}$.

§4. Ultraproducts

In the following, we will consider the necessary and sufficient conditions for a class of structures to be axiomatized by first order logic, that is, be expressed as $\mathrm{Mod}(\mathrm{T})$.

Definition 4.1

A class $\mathcal{K}$ of $\mathcal{L}$-structures is called an **elementary class** if there exists a set T of sentences such that $\mathcal{K} = \mathrm{Mod}(\mathrm{T})$. In this case, we write

$$\mathcal{K} \in \mathrm{EC}_\Delta.$$

To characterize elementary classes, we use a kind of reduced product called "ultraproduct". To define it, we first introduce an ultrafilter.

Definition 4.2

The filter $\mathcal{F}$ on I is an **ultrafilter** (maximal filter) if the following properties are satisfied.

$$\forall X \subset I(X \in \mathcal{F} \vee I - X \in \mathcal{F}).$$

Lemma 4.3

Every filter $\mathcal{F}$ can be expanded to an ultrafilter $\mathcal{U}$.

Proof. Consider the class of all filters including a given filter $\mathcal{F}$.
Since it is closed under the union of chains, by Zorn's lemma, there is a maximal filter $\mathcal{U}$ which is an ultrafilter. □

A principal filter is an ultrafilter. There exists an ultrafilter which is **non-principal**.

Lemma 4.4

There exists a non-principal ultrafilter $\mathcal{U}$ on any infinite set I.

Proof.
Let I be an infinite set, and $\mathcal{F}$ be a Fréchet filter on it (a subset of I whose complement is finite). By the above lemma, an ultrafilter $\mathcal{U}$ can be obtained by expanding $\mathcal{F}$. Then $\mathcal{U}$ is non-principal, since for each $i \in I$, $I - \{i\} \in \mathcal{F} \subseteq \mathcal{U}$, so we have $\{i\} \notin \mathcal{U}$.

Stone's representation theorem

We prove Stone's representation theorem using an ultrafilter.

Theorem 4.8 (in Part 1, Stone's representation theorem)

For any Boolean algebra $\mathfrak{B}$, there exists a set X, and $\mathfrak{B}$ can be embedded in the power set algebra $\mathfrak{P}(X)$. In particular, if $\mathfrak{B}$ is finite, it is isomorphic to $\mathfrak{P}(X)$.

Proof.

- Let $\mathfrak{B} = (B, \vee, \wedge, \neg, 0, 1)$ be a Boolean algebra. Filters, Ultrafilters, and others can naturally be defined for a subset $F \subseteq B$ with the ordering $x \leq y \Leftrightarrow x \wedge y = x$. Let X be the set of all ultrafilters of B and $\mathcal{P}(X)$ be its power set.
- Now, $f : B \to \mathcal{P}(X)$ is defined as follows:
  For each $b \in B$, $f(b)$ is the set of all ultrafilters containing b. We show f is injective.
  If $a \neq b$, then ① $a \wedge (\neg b) \neq 0$ or ② $(\neg a) \wedge b \neq 0$.

  Case ①. Since $\{a, \neg b\}$ has the finite intersection property, it can be extended to an ultrafilter $U \subseteq B$. Thus, $U \in f(a)$ and $U \notin f(b)$, and we have $f(a) \neq f(b)$.
  Case ② can be treated similarly.

- Furthermore, by the property of filter F: $a \wedge b \in F \Leftrightarrow a \in F$ and $b \in F$, we have

$$f(a \wedge b) = f(a) \cap f(b).$$

- Also, by the property of the ultrafilter U: $a \notin U \Leftrightarrow \neg a \in U$, we have

$$f(\neg a) = X - f(a).$$

  Thus, $f : B \to \mathcal{P}(X)$ is embedding.

- If $\mathfrak{B}$ is finite, any ultrafilter must be a principal filter. And its generator is an atom (non-zero minimal element) in $\mathfrak{B}$. So, let X be the set of atoms. It is easy to see that $\mathfrak{B}$ and $\mathfrak{P}(X)$ are isomorphic. □

Ultraproducts

Definition 4.5 (Ultraproducts)

The reduced product $\prod \mathfrak{A}_i/\mathcal{U}$ for an ultrafilter $\mathcal{U}$ is called an **ultraproduct**.

Theorem 4.6 (Łos' theorem)

Let $\mathcal{U}$ be an ultrafilter. For any formula $\varphi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in \prod A_i$,
$\prod \mathfrak{A}_i/\mathcal{U} \models \varphi([a_1], \ldots, [a_n]) \Leftrightarrow \|\varphi(a_1, \ldots, a_n)\| \in \mathcal{U}$.

Proof. By induction on the construction of formulas. The atomic formulas and formulas beginning with $\wedge$ and $\exists$ are treated in Page 17.
Then we only need to treat the case of negation, $\neg\varphi$ since $\vee$ and $\forall$ can be expressed by $\wedge, \exists$ and negation $\neg$.

$$\begin{aligned} \prod \mathfrak{A}_i/\mathcal{U} \models \neg\varphi &\Leftrightarrow \prod \mathfrak{A}_i/\mathcal{U} \not\models \varphi \\ &\Leftrightarrow \|\varphi\| \notin \mathcal{U} \quad (\because \text{induction hypothesis}) \\ &\Leftrightarrow \|\neg\varphi\| \in \mathcal{U} \quad (\because \text{maximality of } \mathcal{U}). \quad \square \end{aligned}$$

By applying the above theorem, we obtain another proof of compactness theorem.

Corollary 4.7 (Compactness theorem)

A theory T has a model iff any finite subset of T has a model.

- The necessity is clear and we show the sufficiency.
- Let I be the set of finite subsets of T. For each $\varphi \in T$, let $J_\varphi = \{i \in I : \varphi \in i\}$. Then $\{J_\varphi : \varphi \in T\}$ has the finite intersection property since $\{\varphi_1, \ldots, \varphi_n\} \in J_{\varphi_1} \cap \cdots \cap J_{\varphi_n}$.
- There exists an ultrafilter $\mathcal{U} \supseteq \{J_\varphi : \varphi \in T\}$ by Lemma 3.3 and Lemma 4.3.
- Let $\mathfrak{A}_i$ be a model for each $i \in I$ and $\mathfrak{A} = \prod \mathfrak{A}_i/\mathcal{U}$. We show that $\mathfrak{A}$ is a model of T.
- First, take an arbitrary $\varphi \in T$. Since

$$i \in J_\varphi \Rightarrow \varphi \in i \Rightarrow \mathfrak{A}_i \models \varphi,$$

  we have $J_\varphi \subseteq \{i : \mathfrak{A}_i \models \varphi\}$. Since $J_\varphi \in \mathcal{U}$, $\|\varphi\| = \{i : \mathfrak{A}_i \models \varphi\} \in \mathcal{U}$.
- By Łos' Theorem, we have $\mathfrak{A} \models \varphi$.

Problem 9

Use ultraproducts to show that any field $\mathcal{F}$ has algebraic closure $\overline{\mathcal{F}}$.

Solution:

- We fix a field $\mathcal{F}$ in a language with constants for their elements.
- Let $\mathcal{F}_P$ be a splitting field of a polynomial P, and for each $Q \in \mathcal{F}[X]$, we put

$$J_Q = \{P \in \mathcal{F}[X] : Q \text{ is splitted into linear factors over } \mathcal{F}_P\}.$$

- Then, $\{J_Q : Q \in \mathcal{F}[X] \text{ and } Q \text{ is not a constant.}\}$ has the finite intersection property $(\because Q_1 \cdots Q_n \in J_{Q_1} \cap \cdots \cap J_{Q_n})$. Therefore, it can be expanded to an ultrafilter $\mathcal{U}$.

- Now consider the ultraproduct $\prod \mathcal{F}_P/\mathcal{U}$, which is a field extension of $\mathcal{F}$.
- For any (non-constant) polynomial $Q \in \mathcal{F}[X]$, the sentence "Q can be splitted over $\mathcal{F}_P$" is true for all P belonging to $J_Q \in \mathcal{U}$, and so it holds in $\prod \mathcal{F}_P/\mathcal{U}$.
- Therefore, $\prod \mathcal{F}_P/\mathcal{U}$ is an algebraically closed field.
- Finally, we define $\overline{\mathcal{F}}$ to be the set of elements of $\prod \mathcal{F}_P/\mathcal{U}$ which is a root of some $P \in \mathcal{F}[X]$. Clearly, $\overline{\mathcal{F}}$ is an algebraic extension of $\mathcal{F}$.
- Now, suppose for the contrary that there is a polynomial in $\overline{\mathcal{F}}[X]$ that has no root in $\overline{\mathcal{F}}$. Then, the root should be to expressed as a root of the polynomial of $\mathcal{F}$ ("Algebraic extension" is transitive), which contradicts with the definition of $\overline{\mathcal{F}}$.
- Therefore, $\overline{\mathcal{F}}$ is an algebraic closure of $\mathcal{F}$.

Theorem 4.8 (Frayne-Morel-Scott)

A class of structures $\mathcal{K}$ is an elementary class (EC_Δ) iff it is closed under elementary equivalences and ultraproduct.

Proof.

- ($\Rightarrow$) is clear and we only show ($\Leftarrow$). Let $\mathcal{K}$ be closed under elementary equivalences and utraproducts. Let $T = \{\sigma : \forall \mathfrak{A} \in \mathcal{K}, \mathfrak{A} \models \sigma\}$ and we claim $\mathcal{K} = \mathrm{Mod}(T)$.
  $\mathcal{K} \subseteq \mathrm{Mod}(T)$ is clear.
  To show $\mathrm{Mod}(T) \subseteq \mathcal{K}$, we let $\mathfrak{B} \in \mathrm{Mod}(T)$.
  Let I be the set of finite subsets of $\mathrm{Th}(\mathfrak{B})$.
- We show that for any $i \in I$, there exists $\mathfrak{A}_i \in \mathcal{K}$ such that $\mathfrak{A}_i \models i$ by the contradiction. Assume some $i \in I$, $\forall \mathfrak{A} \in \mathcal{K}\,(\mathfrak{A} \not\models i)$. Let $i = \{\varphi_1, \ldots, \varphi_n\}$. For any $\mathfrak{A} \in \mathcal{K}$,
  $\mathfrak{A} \models \neg\varphi_1 \vee \cdots \vee \neg\varphi_n$.
  Thus, $\neg\varphi_1 \vee \cdots \vee \neg\varphi_n \in T$. Since $\mathfrak{B} \models T$, we have $\mathfrak{B} \models \neg\varphi_k$ for some $k \in I$.
  This contradicts $\varphi_k \in i \subseteq \mathrm{Th}(\mathfrak{B})$.
- We can construct an ultraproduct of model $\mathfrak{A}$ such that $T = \mathrm{Th}(\mathfrak{B})$ as in the proof of compactness theorem.
- Since $\mathcal{K}$ is closed under ultraproducts, we have $\mathfrak{A} \in \mathcal{K}$.
  Because $\mathcal{K}$ is closed under elementary equivalence, $\mathfrak{A} \equiv \mathfrak{B}$. Thus $\mathfrak{B} \in \mathcal{K}$. □

Definition 4.9

$\prod \mathfrak{A}_i/\mathcal{U}$ is called an **ultrapower** of $\mathfrak{A}$, denoted by $\mathfrak{A}^I/\mathcal{U}$ if $\mathfrak{A}_i = \mathfrak{A}$ for each $i \in I$.

For $a \in |\mathfrak{A}|$, take the function $\lambda i.a \in \prod \mathfrak{A}_i$ that always takes the value a, and let

$$^*a = [\lambda i.a] \in |\mathfrak{A}^I/\mathcal{U}|$$

Then, the function $d : |\mathfrak{A}| \to |\mathfrak{A}^I/\mathcal{U}|$ determined by $d(a) = {}^*a$ is called a **canonical embedding**.

Definition 4.10

An embedding $\phi : \mathfrak{A} \to \mathfrak{B}$ is said to be **elementary** if $\phi(\mathfrak{A}) \prec \mathfrak{B}$.

Theorem 4.11

Let $\prod \mathfrak{A}^I/\mathcal{U}$ be an ultrapower of $\mathfrak{A}$. Then the canonical embedding $d : |\mathfrak{A}| \to |\mathfrak{A}^I/\mathcal{U}|$ is elementary. In particular, $\mathfrak{A} \equiv \mathfrak{A}^I/\mathcal{U}$.

Proof. For any formula $\varphi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in |\mathfrak{A}|$, by Łos theorem,

$$\mathfrak{A}^I/\mathcal{U} \models \varphi({}^*a_1, \ldots, {}^*a_n) \Leftrightarrow \{i \in I : \mathfrak{A} \models \varphi(a_1, \ldots, a_n)\} \in \mathcal{U}$$
$$\Leftrightarrow \mathfrak{A} \models \varphi(a_1, \ldots, a_n).$$

Thus d is an elementary embedding. $d(\mathfrak{A}) \cong \mathfrak{A}$ so $\mathfrak{A} \equiv \mathfrak{A}^I\mathcal{U}$ is obvious. □

Theorem 4.12 (Keisler-Shelah)

$\mathfrak{A} \equiv \mathfrak{B} \Leftrightarrow$ There exists some I and a ultraproduct $\mathcal{U}$ such that $\mathfrak{A}^I/\mathcal{U} \cong \mathfrak{B}^I/\mathcal{U}$.

Proof. $(\Leftarrow)$ is derived from the last theorem. The proof of $(\Rightarrow)$ is too technical, see *Model Theory: Third Edition - C.C. Chang, H. Jerome Keisler* for details. □

Assuming Keisler-Shelah theorem, we obtain the following.

Corollary 4.13

The structural class $\mathcal{K}$ is the elementary class (EC_Δ) iff the following two conditions hold.

(1) $\mathcal{K}$ is closed under ultraproducts and isomorphisms.

(2) $\mathfrak{A}^I/\mathcal{U} \in \mathcal{K} \Rightarrow \mathfrak{A} \in \mathcal{K}$. (It is closed under inverses of ultrapower).

Proof.

- $\Rightarrow$ is obvious.
- To show $\Leftarrow$, assume (1), (2) and prove $\mathcal{K}$ is closed by elementary equivalence. So, let $\mathfrak{A} \equiv \mathfrak{B}$ and $\mathfrak{A} \in \mathcal{K}$. We want to show $\mathfrak{B} \in \mathcal{K}$.
- By the Keisler-Shelah theorem, there exists a ultraproduct $\mathcal{U}$ on I such that $\mathfrak{A}^I/\mathcal{U} \cong \mathfrak{B}^I/\mathcal{U}$.
- Since $\mathcal{K}$ is closed under ultraproducts, $\mathfrak{A}^I/\mathcal{U} \in \mathcal{K}$.
- Since $\mathcal{K}$ is closed under isomorphisms, $\mathfrak{B}^I/\mathcal{U} \in \mathcal{K}$.
- Since $\mathcal{K}$ is closed under inverses of ultrapower, we have $\mathfrak{B} \in \mathcal{K}$.

$\square$

- The class $\mathcal{K}$ of $\mathcal{L}$-structures is called a **projective class** or **pseudo-elementary class**, denote PC_Δ, if there exists an elementary class $\mathcal{K}' \in \mathrm{EC}_\Delta$ in an extended language $\mathcal{L}' \supseteq \mathcal{L}$ such that $\mathcal{K}$ equals the class of $\mathcal{L}$-reduced structures in $\mathcal{K}'$.
- The class of orderable groups is a projective class.
- It is easy to see that PC_Δ is also closed under ultraproducts and isomorphisms. Various characterizations are also known for PC_Δ.
- The following result is particularly useful for applications such as a proof of Craig's interpolation theorem.

Theorem 4.14

If $\mathcal{K}, \mathcal{K}' \in \mathrm{PC}_\Delta$ and $\mathcal{K} \cap \mathcal{K}' = \varnothing$, then there exists $\mathcal{J} \in \mathrm{EC}$ such that $\mathcal{K} \subseteq \mathcal{J}$ and $\mathcal{J} \cap \mathcal{K}' = \varnothing$, where $\mathcal{J} \in \mathrm{EC}$ means that $\mathcal{J} = \mathrm{Mod}(\{\sigma\})$ with a single sentence σ.

Thank you for your attention!