Topics in Applied Math: Logic and Foundations of Mathematics

Part 3. Basic Model theory

Kazuyuki Tanaka

BIMSA

October 15, 2025

清華大学 求真书院
Qiuzhen College, Tsinghua University

Logic and Foundations

- **Part 1. Equational theory**
- **Part 2. First order theory**
- **Part 3. Basic Model theory**
- **Part 4. First order arithmetic and incompleteness theorems**
- **Part 5. Models of first-order arithmetic**
- **Part 6. Second order arithmetic and reverse mathematics**

Part 3. Schedule

- Oct. 15, (1) Various forms of axiomatic systems and reduced products
- Oct. 17, (2) Ultra products and non-standard analysis

Interpretations (in Part 2)

Definition 4.6 (language interpretation)

Given two languages $\mathcal{L}, \mathcal{L}'$ and a theory T' in the language $\mathcal{L}'$. A pair $\langle U, I \rangle$ that satisfies the following is called a **interpretation (translation)** of language $\mathcal{L}$ (in T').

(1) $U(x)$ is a formula in $\mathcal{L}'$. $T' \vdash \exists x U(x)$. (It represents the domain of a theory.)

(2) I is a function from $\mathcal{L}$ to formulas in $\mathcal{L}'$.
If $\mathtt{f}$ is an n-ary function symbol, $I(\mathtt{f})$ is an $(n+1)$-ary formula and

$$T' \vdash \forall x_1 \cdots \forall x_n (U(x_1) \land \cdots \land U(x_n) \to \exists! y (I(\mathtt{f})(x_1, \ldots, x_n, y) \land U(y))).$$

If R is an n-ary relation symbol, $I(\mathrm{R})$ is also an n-ary formula.

Interpretation of formulas

We interpret an $\mathcal{L}$-formula φ into a formula φ^I in $\mathcal{L}'$.

- We first modify $I(\mathtt{f})$ as the following $I'(\mathtt{f})$ to define a function. With a new constant a, we set
$$I'(\mathtt{f})(x_1,\dots,x_n,y) \Leftrightarrow$$
$$((U(x_1)\wedge\cdots\wedge U(x_n))\wedge I(\mathtt{f})(x_1,\dots,x_n,y))\vee((\neg U(x_1)\vee\cdots\vee\neg U(x_n))\wedge y=a).$$
Then, $I'(\mathtt{f})$ defines a function, denoted f^I or simply f.
- Next, $I(\mathrm{R})$ define a relation R^I, also denoted R. Then, terms and atomic formulas of $\mathcal{L}$ will remain unchanged after interpretation. The propositional connectives are also kept unchanged.
- We only need to deal with quantifiers.
(1) $(\exists x\psi)^I$ is $\exists x(U(x)\wedge\psi^I)$. (2) $(\forall x\psi)^I$ is $\forall x(U(x)\to\psi^I)$.

Interpretation of theories

Definition 4.7

- Let T and T' be theories in languages $\mathcal{L}$ and $\mathcal{L}'$, respectively. Suppose that $\langle U, I \rangle$ is an interpretation of language $\mathcal{L}$ in T'.
- Then, $\langle U, I \rangle$ is called an **interpretation of theory** T in T', if for any sentence σ in $\mathcal{L}$,

$$T \vdash \sigma \quad \Rightarrow \quad T' \vdash \sigma^I.$$

- If there is an interpretation of T in T', T is said to be **interpretable** within T'.
- Moreover, $\langle U, I \rangle$ is called a **faithful interpretation** of T' in T, if the following holds

$$T \vdash \sigma \quad \Leftrightarrow \quad T' \vdash \sigma^I.$$

Hints to HW

Problem 5 (HW # 2)

1. Show that Peano arithmetic PA is interpretable within ZF set theory.
2. Show that ZF without Infinity axiom is interpretable within PA.

Hints:

(1) In ZF, $U(x)$ is defined as a predicate representing a finite ordinal, and the arithmetic operations of PA are the same as those for ordinals.

- To show that this is a proper interpretation of PA in ZF, it is sufficient to show that the interpretation of mathematical induction is provable in ZF.

(2) In PA, let $U(x)$ be $x = x$, and define $I(k \in n)$ to denote that the $k+1$-th digit of the binary expression of n is 1.

- Intuitively, $k \in n$ expresses that the set with code k belongs to the set with code n. By such an interpretation, all the axioms of ZF other than the axiom of infinity are provable within PA. In particular, the axiom of replacement is interpreted into a collection principle (a variation of induction).

Introduction to Part 3

- In part 1, we gave the necessary and sufficient conditions for a class of structures to be axiomatized in equational class theory (Birkhoff's variety theorem).
- In this part, we will discuss various forms of axiomatic systems in first-order logic (e.g., Horn theory) and the properties of the models of such systems. To study them, we will introduce the basic concepts of model theory, such as elementary substructures, elementary class, model-complete, reduced product and ultraproduct.
- In addition, we will discuss non-standard analysis as an important application of model theory. Using ultrapower or utraproduct, we can construct a nonstandard extension of real numbers including infinitesimals and infinities, where the limit can be replaced by a finite calculation.
- The non-standard analysis we introduce here will be adopted with some restrictions in a weaker system (to be discussed in the part of reverse mathematics).

Definition 1.1 (Similar to the definitions for algebraic structures)

Let $\mathfrak{A}$ and $\mathfrak{B}$ be $\mathcal{L}$-structures. A morphism $\phi : |\mathfrak{A}| \to |\mathfrak{B}|$ is called an **homomorphism** denoted $\phi : \mathfrak{A} \to \mathfrak{B}$ if the following holds:

- for all $\mathtt{f} \in \mathcal{L}$ and $a_0, \ldots, a_{m-1} \in |\mathfrak{A}|$,

$$\phi(\mathtt{f}^{\mathfrak{A}}(a_0, \ldots, a_{m-1})) = \mathtt{f}^{\mathfrak{B}}(\phi(a_0), \ldots, \phi(a_{m-1})),$$

- for all $\mathrm{R} \in \mathcal{L}$ and $a_0, \ldots, a_{n-1} \in |\mathfrak{A}|$,

$$\mathrm{R}^{\mathfrak{A}}(a_0, \ldots, a_{n-1}) \Rightarrow \mathrm{R}^{\mathfrak{B}}(\phi(a_0), \ldots, \phi(a_{n-1})).$$

A homomorphism ϕ is called an **embedding** if it is injective (one-to-one) and for all R and $a_0, \ldots, a_{n-1} \in |\mathfrak{A}|$,

$$\mathrm{R}^{\mathfrak{A}}(a_0, \ldots, a_{n-1}) \Leftrightarrow \mathrm{R}^{\mathfrak{B}}(\phi(a_0), \ldots, \phi(a_{n-1})).$$

If an embedding $\phi : \mathfrak{A} \to \mathfrak{B}$ is surjective, we say that ϕ is an **isomorphism**. In this case, we also say that $\mathfrak{A}$ and $\mathfrak{B}$ are **isomorphic** and write

$$\mathfrak{A} \cong \mathfrak{B}$$

The set of atomic sentences and negations of atomic sentences contained in $\mathrm{Th}(\mathfrak{A}_A)$ is the diagram $\mathrm{Diag}(\mathfrak{A})$ of $\mathfrak{A}$. (Definition 2.1.7). Using this, the following is straightforward:

Lemma 1.2

Let $\mathfrak{A}$ and $\mathfrak{B}$ be $\mathcal{L}$-structures. A morphism $\phi : |\mathfrak{A}| \to |\mathfrak{B}|$ is an embedding iff $\mathfrak{B}_{\phi(A)}$ is a model of $\mathrm{Diag}(\mathfrak{A})$.

Definition 1.3

An $\mathcal{L}$-structure $\mathfrak{A}$ is a **substructure** of an $\mathcal{L}$-structure $\mathfrak{B}$ if $|\mathfrak{A}| \subseteq |\mathfrak{B}|$ and the identity map $i_{|\mathfrak{A}|} : |\mathfrak{A}| \to |\mathfrak{B}|$ is an embedding (i.e., if the interpretation of a symbol $\mathtt{f}$ or R in the structure $\mathfrak{A}$ is the same as the interpretation of the corresponding symbol in $\mathfrak{B}$ restricted to $\mathfrak{A}$). Then we write

$$\mathfrak{A} \subseteq \mathfrak{B}.$$

Example 1

$$(\mathbb{N}, +, \cdot, 0, 1, <) \subseteq (\mathbb{R}, +, \cdot, 0, 1, <).$$

For a homomorphism $\phi : \mathfrak{A} \to \mathfrak{B}$, the substructure of $\mathfrak{B}$ whose domain is the image $\phi(|\mathfrak{A}|)$ is called the **homomorphic image**, and written as $\phi(\mathfrak{A})$. For an embedding $\phi : \mathfrak{A} \to \mathfrak{B}$, $\phi(\mathfrak{A})$ and $\mathfrak{A}$ are isomorphic.

The logical counterpart of isomorphism is "**elementary equivalence**." "Elementary" is a term used almost synonymously with "first-order logic" by the Tarski school at Berkeley.

Definition 1.4

$\mathcal{L}$-structures $\mathfrak{A}$ and $\mathfrak{B}$ are **elementary equivalent**, denoted $\mathfrak{A} \equiv \mathfrak{B}$, if the same $\mathcal{L}$-sentences hold in both structures, that is, $\mathrm{Th}(\mathfrak{A}) = \mathrm{Th}(\mathfrak{B})$

Lemma 1.5

Any two isomorphic structures are elementary equivalent, that is,

$$\mathfrak{A} \cong \mathfrak{B} \Rightarrow \mathfrak{A} \equiv \mathfrak{B}$$

It must be easier to show that for an isomorphism $\phi : \mathfrak{A} \to \mathfrak{B}$, $\mathrm{Th}(\mathfrak{A}_A) = \mathrm{Th}(\mathfrak{B}_{\phi(A)})$.

Definition 1.6

A structure $\mathfrak{A}$ is an **elementary substructure** of a structure $\mathfrak{B}$, denoted $\mathfrak{A} \prec \mathfrak{B}$, if $\mathfrak{A}$ is a substructure of $\mathfrak{B}$ and the same $\mathcal{L}_A$-sentences hold in both structures, i.e, $\mathrm{Th}(\mathfrak{A}_A) = \mathrm{Th}(\mathfrak{B}_A)$.

Note that the notion of elementary substructure is stronger than that of elementary equivalence:

$$\mathfrak{A} \cong \mathfrak{B} \Rightarrow \mathfrak{A} \prec \mathfrak{B} \Rightarrow \mathfrak{A} \equiv \mathfrak{B}$$

Also, $\prec$ is a transitive relation.

Example 2

Let $\mathfrak{N}_< = (\mathbb{N}, <)$. Then $\mathfrak{N}_<^+ = (\mathbb{N} - \{0\}, <)$.

$$\mathfrak{N}_< \cong \mathfrak{N}_<^+, \quad \mathfrak{N}_< \equiv \mathfrak{N}_<^+, \quad \mathfrak{N}_<^+ \subseteq \mathfrak{N}_<.$$

but,

$$\mathfrak{N}_<^+ \not\prec \mathfrak{N}_<.$$

Problem 2

Let $\mathfrak{Q}_< = (\mathbb{Q}, <)$, $\mathfrak{R}_< = (\mathbb{R}, <)$. Which of $\cong$, $\subseteq$, $\equiv$, $\prec$ hold between them?

Answer. $\mathfrak{Q}_< \subseteq \mathfrak{R}_<$, $\mathfrak{Q}_< \equiv \mathfrak{R}_<$ and $\mathfrak{Q}_< \prec \mathfrak{R}_<$, but $\mathfrak{Q}_< \not\cong \mathfrak{R}_<$.

Problem 3

Suppose $\mathfrak{A} \equiv \mathfrak{B}$ and $|\mathfrak{A}|$ is finite. Show that $\mathfrak{A} \cong \mathfrak{B}$.

Theorem 1.7 (Tarski-Vaught's criterion)

$\mathfrak{A} \prec \mathfrak{B}$ iff $\mathfrak{A} \subseteq \mathfrak{B}$ and for any formula $\varphi(x, y_1, \ldots, y_m)$ and any $a_1, \ldots, a_m \in A$,

$$\mathfrak{B}_A \models \exists x \varphi(x, a_1, \ldots, a_m) \Rightarrow \text{there exists an } a \in |\mathfrak{A}| \text{ s.t. } \mathfrak{B}_A \models \varphi(a, a_1, \ldots, a_m).$$

Proof. (only if) is clear.

For (if), we will show that for any formula $\varphi(x_1, \ldots, x_n)$ and any $a_1, \ldots, a_n \in A$,

$$\mathfrak{A}_A \models \varphi(a_1, \ldots, a_n) \Leftrightarrow \mathfrak{B}_A \models \varphi(a_1, \ldots, a_n)$$

by induction on the construction of φ.

If φ is an atomic formula, then the equivalence $\Leftrightarrow$ holds since $\mathfrak{A} \subseteq \mathfrak{B}$.

For the induction step, the essential case is that φ is of the form $\exists x \psi$.
$\Leftarrow$ can be obtained from the condition of the theorem and $\Rightarrow$ is trivial. □

Problem 4.1

For a theory T of a language $\mathcal{L}$, the following are equivalent:

(i) (**Weak Henkin property**). For an $\mathcal{L}$-formula $\varphi(x)$ with no other free variable than x, there are $\mathcal{L}$-terms $t_1, \ldots, t_n$ that do not contain variables,

$$T \vdash \exists x \varphi(x) \to \varphi(t_1) \vee \cdots \vee \varphi(t_n).$$

(ii) For any model $\mathfrak{A}$ of T, its smallest substructure is an elementary substructure.

Problem 4.2

For a theory T of a language $\mathcal{L}$, the following are equivalent:

(i) (**Weak Skolem property**). For any $\mathcal{L}$-formula $\varphi(x, y_1, \ldots, y_m)$ with no other free variables than displayed, there exist terms $t_1(y_1, \ldots, y_m), \ldots, t_n(y_1, \ldots, y_m)$ such that

$$T \vdash \exists x \varphi(x, y_1, \ldots, y_m) \to \varphi(t_1(y_1, \ldots, y_m), y_1, \ldots, y_m) \vee \cdots \\ \vee \varphi(t_n(y_1, \ldots, y_m), y_1, \ldots, y_m).$$

(ii) For any model $\mathfrak{A}$ of T, any substructure becomes an elementary substructure.

Solution:

(i) $\to$ (ii) of problem 4.1

- Let $\mathfrak{A}$ a model of T, and $\mathfrak{B}$ be its smallest substructure. Then $|\mathfrak{B}|$ is the set of $t^{\mathfrak{A}}$ where t is a term without variables.
- To applying the Tarski-Vaught test, assume $\mathfrak{A} \models \exists x \varphi(x)$. Note that $\exists x \varphi(x)$ may include elements of $\mathfrak{B}$ as closed terms. Then by the weak Henkin property of T, there exist closed terms $t_1, \ldots, t_n$ such that $\mathfrak{A} \models \varphi(t_1) \vee \cdots \vee \varphi(t_n)$. So, there is $i \leq n$ such that $\mathfrak{A} \models \varphi(t_i)$ with $t_i \in |\mathfrak{B}|$.
- Hence, by the Tarski-Vaught criterion, $\mathfrak{B} \prec \mathfrak{A}$.

(ii) $\to$ (i) of problem 4.1

- By using contraposition, let T be a theory without the weak Henkin property. Then, there exists a formula $\varphi(x)$ and $T \cup \{\exists x \varphi(x)\} \cup \{\neg\varphi(t) : t$ is a term without variables$\}$ is consistent (by compactness). Let $\mathfrak{A}$ be a model of such a theory and $\mathfrak{B}$ be its smallest substructure consisting of closed terms.
- Assume that $\mathfrak{B} \prec \mathfrak{A}$. So, since $\mathfrak{B}$ is also a model of such a theory, $\mathfrak{B} \models \exists x \varphi(x)$, and then there is a closed term t such that $\mathfrak{B} \models \varphi(t)$. Again, since $\mathfrak{B} \prec \mathfrak{A}$, we have $\mathfrak{A} \models \varphi(t)$, which is a contradiction.

Problem 4.2 can be solved by similar argument.

Definition 1.8

A countable ascending sequence of structures

$$\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \cdots \subseteq \mathfrak{A}_i \subseteq \cdots (i < \omega)$$

is called a **chain** of structures. Then, the structure $\mathfrak{A} = \bigcup_{i<\omega} \mathfrak{A}_i$, which is naturally defined as the limit of the chain, is called the **union** of the chain. Note that it is clear that for each $i < \omega$, $\mathfrak{A}_i$ is a substructure of $\mathfrak{A}$.

Definition 1.9

A chain of structures is called an **elementary chain** if

$$\mathfrak{A}_0 \prec \mathfrak{A}_1 \prec \cdots \prec \mathfrak{A}_i \prec \cdots \quad (i < \omega)$$

And the structure $\mathfrak{A} = \bigcup_{i<\omega} \mathfrak{A}_i$ is called the **union** of the elementary chain.

Theorem 1.10 (**Elementary chain theorem**)

Let $\mathfrak{A}_0 \prec \mathfrak{A}_1 \prec \cdots$ be an elementary chain. Let $\mathfrak{A}$ be the union of the elementary chain. Then for each i, $\mathfrak{A}_i \prec \mathfrak{A}$.

Proof. First, let $\mathfrak{A}$ be the union of the elementary chains $\mathfrak{A}_0 \prec \mathfrak{A}_1 \prec \cdots$. We claim that for all i and for any $a_1, \ldots, a_n \in A_i$,

$$\mathfrak{A}_{iA_i} \models \varphi(a_1, \ldots, a_n) \Leftrightarrow \mathfrak{A}_{A_i} \models \varphi(a_1, \ldots, a_n).$$

We prove the claim by induction on the construction of the formula $\varphi(x_1, \ldots, x_n)$, based on Tarski-Vaught's criterion. Note that if the induction is carried out for each fixed i, the argument does not work well.

- If $\varphi(x_1, \ldots, x_n)$ is an atomic formula, the claim is obvious.
- The essential step in induction is the case that $\varphi \equiv \exists x \psi$. Now, take any i and any $a_1, \ldots, a_n \in A_i$.
  ($\Rightarrow$) follows immediately from the induction hypothesis.
  To show ($\Leftarrow$), assume that $\mathfrak{A}_{A_i} \models \exists x \varphi(x, a_1, \ldots, a_m)$.
  Then for some $a \in A$, $\mathfrak{A}_A \models \varphi(a, a_1, \ldots, a_m)$. Take a sufficiently large $j \geq i$ such that $a, a_1, \ldots, a_m \in A_j$. So, we have $\mathfrak{A}_{A_j} \models \varphi(a, a_1, \ldots, a_m)$.
  By the induction hypothesis, we have $\mathfrak{A}_{jA_j} \models \varphi(a, a_1, \ldots, a_m)$. Therefore, $\mathfrak{A}_{jA_j} \models \exists x \varphi(x, a_1, \ldots, a_m)$. Finally, since $\mathfrak{A}_i \prec \mathfrak{A}_j$ by transitivity of $\prec$, we have $\mathfrak{A}_{iA_i} \models \exists x \varphi(x, a_1, \ldots, a_m)$.

□

It is easy to generalize the above theorem to transfinite sequences for any ordinal α.

$$\mathfrak{A}_0 \prec \mathfrak{A}_1 \prec \cdots \prec \mathfrak{A}_\beta \prec \cdots \quad (\beta < \alpha)$$

∀-theory and ∀∃-theory

We will extend Birkhoff's equational class theorem to ∀-theories and ∀∃-theories, which are the most commonly used forms of axiomatic systems in mathematics.

Definition 2.1

Let T be a theory of a language $\mathcal{L}$. We denote the class of all models of T by $\mathrm{Mod(T)}$, i.e.,

$$\mathrm{Mod(T)} = \{\mathfrak{A} : \mathfrak{A} \models \mathrm{T}\}$$

Birkhoff's equational class theorem can be restated as the following extended form:

Theorem 2.2 (Birkhoff's Theorem 1.3.13)

Let $\mathcal{K}$ be a class of $\mathcal{L}$-structures, then the following are equivalent:

(1) $\mathcal{K}$ is closed under direct products, substructures and homomorphic images.

(2) There exists a theory T consisting of (the universal closures of) atomic formulas such that $\mathcal{K} = \mathrm{Mod}(T)$.

Definition 2.3

Let $\mathfrak{A}_i$ $(i \in I)$ be $\mathcal{L}$-structures. The structure $\prod \mathfrak{A}_i = (\prod A_i, \mathtt{f}^{\prod \mathfrak{A}_i}, \ldots, \mathrm{R}^{\prod \mathfrak{A}_i}, \ldots)$, where

$$\mathtt{f}^{\prod \mathfrak{A}_i}(a_1, \ldots, a_n) = \lambda i.\mathtt{f}^{\mathfrak{A}_i}(a_1(i), \ldots, a_n(i)), \quad \text{for all } a_1, \ldots, a_n \in \prod A_i,$$

$$\mathrm{R}^{\prod \mathfrak{A}_i}(a_1, \ldots, a_n) \Leftrightarrow \forall i \in I\,(\mathrm{R}^{\mathfrak{A}_i}(a_1(i), \ldots, a_n(i))), \quad \text{for all } a_1, \ldots, a_n \in \prod A_i,$$

is called their **direct product**. Here, $\lambda i.y_i$ represents a function $f(i) = y_i$ for each i.

A question naturally arises from Birkhoff's theorem. If we consider the closure conditions of $\mathcal{K}$ separately, what kind of axiomatic system corresponds to each case?

(1) Direct products will be generalized to “reduced products” and studied today.
(2) Substructures will be also discussed in this lecture.
(3) As for homomorphic images, the following nice theorem will be shown in Part 5.

Theorem 2.4 (Lyndon theorem)

The following are equivalent.

1. $\mathrm{Mod(T)}$ is closed under homomorphic images.
2. There exists a theory T' of sentences expressed without negation $\neg$ such that $\mathrm{Mod(T)} = \mathrm{Mod(T')}$.

Definition 2.5

For an open formula (a formula without quantifiers) φ,

$\forall x_1 \cdots \forall x_m \varphi$ is called a ∀ **formula** (or **universal**, Π_1), and

$\forall x_1 \cdots \forall x_n \exists y_1 \cdots \exists y_m \varphi$ is called a ∀∃ **formula** (or **universal-existential**, Π_2).

A set of ∀ sentences is called a **∀-theory** or a **universal theory**, and a set of ∀∃ sentences is called a **∀∃-theory** or an **inductive theory**.

Theorem 2.6 (Łoś-Tarski)

The following two conditions are equivalent.

1. $\mathrm{Mod(T)}$ is closed under substructures.
2. There exists an ∀-theory T' such that $\mathrm{Mod(T)} = \mathrm{Mod(T')}$.

Proof.

To show $(2) \Rightarrow (1)$, let T be a ∀-theory and $\mathfrak{B} \subseteq \mathfrak{A} \in \mathrm{Mod(T)}$.

- We want to show $\mathfrak{B} \in \mathrm{Mod(T)}$.
- Take any $\forall x_1 \cdots x_n \varphi(x_1, \ldots, x_n) \in T$. Then, $\mathfrak{A} \models \forall x_1 \cdots x_n \varphi(x_1, \ldots, x_n)$.
- So, for any $b_1, \ldots, b_n \in B \subseteq A$, $\mathfrak{A}_B \models \varphi(b_1, \ldots, b_n)$. Since φ is an open formula, from $\mathfrak{B} \subseteq \mathfrak{A}$ we have $\mathfrak{B}_B \models \varphi(b_1, \ldots, b_n)$.
- Thus, $\mathfrak{B} \models \forall x_1 \cdots \forall x_n \varphi(x_1, \ldots, x_n)$ and so $\mathfrak{B} \in \mathrm{Mod(T)}$ as required.

To show $(1) \Rightarrow (2)$, suppose $\mathrm{Mod(T)}$ is closed under substructures.

- Let $T' = \{\sigma : \sigma$ is a ∀-sentence, and $T \vdash \sigma\}$.
- Since $\mathrm{Mod(T)} \subseteq \mathrm{Mod(T')}$ is obvious, it is sufficient to show $\mathrm{Mod(T')} \subseteq \mathrm{Mod(T)}$.
- Take an $\mathfrak{A} \in \mathrm{Mod(T')}$. Let $D = \mathrm{Diag}(\mathfrak{A})$ (the basic diagram). If $D \cup T$ has a model $\mathfrak{B}_A$, then it clearly contains a substructure isomorphic to $\mathfrak{A}$. So by the assumption, $\mathfrak{A} \in \mathrm{Mod(T)}$.
- Hence, it is sufficient to show that $D \cup T$ has a model.

Claim: $D \cup T$ has a model.

- By way of contradiction, assume that $D \cup T$ has no model. Then, there exists a finite set $\{\varphi_1(a_1, \ldots, a_n), \ldots, \varphi_k(a_1, \ldots, a_n)\} \subset D$ such that the following is inconsistent.

$$\{\varphi_1(a_1, \ldots, a_n), \ldots, \varphi_k(a_1, \ldots, a_n)\} \cup T$$

- Let $\varphi(a_1, \ldots, a_n) = \varphi_1(a_1, \ldots, a_n) \wedge \cdots \wedge \varphi_k(a_1, \ldots, a_n)$. Then, we have,

$$T \vdash \neg\varphi(a_1, \ldots, a_n).$$

- Since T does not contain the constants $a_1, \ldots, a_n$, we replace them with variables $x_1, \ldots, x_n$, and we have $T \vdash \neg\varphi(x_1, \ldots, x_n)$, so $T \vdash \forall x_1 \cdots \forall x_n \neg\varphi(x_1, \ldots, x_n)$.
- Since $\forall x_1 \cdots \forall x_n \neg\varphi(x_1, \ldots, x_n) \in T'$, we have $\mathfrak{A} \models \forall x_1 \cdots \forall x_n \neg\varphi(x_1, \ldots, x_n)$, that is, $\mathfrak{A}_A \models \neg\varphi(a_1, \ldots, a_n)$. This implies that for some $i \leq k$, $\mathfrak{A}_A \models \neg\varphi_i(a_1, \ldots, a_n)$, and so $\neg\varphi_i(a_1, \ldots, a_n)$ belongs to D, which is a contradiction. □

Problem 5

Consider why the above theorem cannot be rephrased as follows.
If $\mathcal{K}$ is a class of $\mathcal{L}$ structure, the following two are equivalent:

(1) $\mathcal{K}$ is closed under substructures.
(2) There exists a ∀-Theory T such that $\mathcal{K} = \mathrm{Mod}(T)$.

Theorem 2.7 (Chang-Łoś-Suszko)

The followings are equivalent.

(1) $\mathrm{Mod(T)}$ is closed under the union of chains. That is, if $\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \cdots$ is a chain of models of T, the union is also a model of T.

(2) There exists a ∀∃-theory T' such that $\mathrm{Mod(T')} = \mathrm{Mod(T)}$.

Proof.

To show (2) ⇒ (1), Let $\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \cdots$ be a chain of models of ∀∃ theory T'.
We want to show $\mathfrak{A} = \bigcup_{i \in \mathbb{N}} \mathfrak{A}_i$ is also a model of T'.

- Let $\forall x_1 \cdots \forall x_n \exists y_1 \cdots \exists y_m \varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ be a ∀∃-sentence in T'.
- Take any $a_1, \ldots, a_n \in A$. Then there exists some k such that $\{a_1, \ldots, a_n\} \subseteq A_k$.
- Since $\mathfrak{A}_k$ is a model of T', $\mathfrak{A}_{kA_k} \models \exists y_1 \cdots \exists y_m \varphi(a_1, \ldots, a_n, y_1, \ldots, y_m)$, and so there exist $b_1, \ldots, b_m \in A_k$ such that $\mathfrak{A}_{kA_k} \models \varphi(a_1, \ldots, a_n, b_1, \ldots, b_m)$.
- Since φ is open and $\mathfrak{A}_k \subseteq \mathfrak{A}$, we have $\mathfrak{A}_A \models \varphi(a_1, \ldots, a_n, b_1, \ldots, b_m)$.
- Thus $\mathfrak{A}_A \models \exists y_1 \cdots \exists y_m \varphi(a_1, \ldots, a_n, y_1, \ldots, y_m)$. Since $a_1, \ldots, a_n \in A$ are arbitrary,

$$\mathfrak{A} \models \forall x_1 \cdots \forall x_n \exists y_1 \cdots \exists y_m \varphi(x_1, \ldots, x_n, y_1, \ldots, y_m).$$

To show (1) $\Rightarrow$ (2), Let $T' = \{\sigma : \sigma$ is a $\forall\exists$-sentences, where $T \vdash \sigma\}$.

- Since $\mathrm{Mod(T)} \subseteq \mathrm{Mod(T')}$ is obvious, it suffices to show that $\mathrm{Mod(T')} \subseteq \mathrm{Mod(T)}$.
- Let $\mathfrak{A} \in \mathrm{Mod(T')}$ and $D_\forall$ be the set of all $\forall$-sentence contained in $\mathrm{Th}(\mathfrak{A}_\mathrm{A})$.
  - Using the compactness theorem in the same way as the proof of Łoś-Tarski theorem, $D_\forall \cup T$ has a model $\mathfrak{B}_A$.
  - Then $\mathfrak{B}$ as a reduct of $\mathfrak{B}_A$ is also a model of T, and so we have $\mathfrak{A} \subseteq \mathfrak{B}$
- Next let $D = \mathrm{Diag}(\mathfrak{B})$. We want to show $D \cup \mathrm{Th}(\mathfrak{A}_\mathrm{A})$ has a model.
  - Conversely, assume there is a conjunction φ of sentences from D s.t. $\mathrm{Th}(\mathfrak{A}_A) \vdash \neg\varphi$.
  - Assuming $b_1, \ldots, b_n$ are the constants appearing φ belonging to $B - A$, we express φ by $\varphi(b_1, \ldots, b_n)$, that is, $\varphi(x_1, \ldots, x_n)$ is a formula in language $\mathcal{L}_A$.
  - Since $\mathrm{Th}(\mathfrak{A}_A) \vdash \neg\varphi(b_1, \ldots, b_n)$, we have

$$\mathrm{Th}(\mathfrak{A}_A) \vdash \forall x_1 \cdots \forall x_n \neg\varphi(x_1, \ldots, x_n) \;\Rightarrow\; \forall x_1 \cdots \forall x_n \neg\varphi(x_1, \ldots, x_n) \in D_\forall$$

$$\Rightarrow\; \mathfrak{B}_A \models \forall x_1 \cdots \forall x_n \neg\varphi(x_1, \ldots, x_n) \;\Rightarrow\; \mathfrak{B}_A \models \neg\varphi(b_1, \ldots, b_n) \;\Rightarrow\; \text{Contradiction.}$$

- Now let $\mathfrak{A}'_A$ be the model of $D \cup \mathrm{Th}(\mathfrak{A}_\mathrm{A})$. Then $\mathfrak{B} \subseteq \mathfrak{A}'$ and $\mathfrak{A} \prec \mathfrak{A}'$.

- To summarize, for $\mathfrak{A} \in \mathrm{Mod}(\mathrm{T}')$, there is a model $\mathfrak{B}(\supseteq \mathfrak{A})$ of T, and also a $\mathfrak{A}' \supseteq \mathfrak{B}$ such that $\mathfrak{A} \prec \mathfrak{A}'$.
- Since $\mathfrak{A}' \in \mathrm{Mod}(\mathrm{T}')$, we can use the same argument to find a model $\mathfrak{B}'$ of T such that $\mathfrak{B}' \supseteq \mathfrak{A}'$, and there is a $\mathfrak{A}'' \supseteq \mathfrak{B}'$ such that $\mathfrak{A}' \prec \mathfrak{A}''$.
- By repeating this process, we have an elementary chain of models of T'

$$\mathfrak{A} \prec \mathfrak{A}' \prec \mathfrak{A}'' \prec \cdots$$

and a chain of models of T

$$\mathfrak{B} \subseteq \mathfrak{B}' \subseteq \mathfrak{B}'' \subseteq \cdots .$$

Since $\mathfrak{A}^{(n)} \subseteq \mathfrak{B}^{(n)} \subseteq \mathfrak{A}^{(n+1)}$, the two chains have the same union, denoted $\mathfrak{A}_\infty$.

- On one hand, by the elementary chain theorem, $\mathfrak{A} \prec \mathfrak{A}_\infty$.
- On the other hand, $\mathfrak{A}_\infty \models T$ by condition (1) of the theorem.

- Therefore, $\mathfrak{A}$ is a model of T, as desired. □

Problem 6

Let T be a $\forall\exists$ theory, and φ_1, φ_2 be $\forall\exists$ sentences. Now, suppose any model $\mathfrak{A}$ of T can be extended to a model of $T \cup \{\varphi_1\}$ and a model of $T \cup \{\varphi_2\}$. Then show that any model $\mathfrak{A}$ of T can be extended to $T \cup \{\varphi_1, \varphi_2\}$.

Solution:

- Construct a chain $\mathfrak{A} \subseteq \mathfrak{A}_1 \subseteq \mathfrak{A}_2 \subseteq \cdots$ of a $\forall\exists$ theory T such that $\mathfrak{A}_{2i+1}$ is a model of $T \cup \{\varphi_1\}$ and $\mathfrak{A}_{2i+2}$ is a model of $T \cup \{\varphi_2\}$.
- Since $\bigcup_{i\in\mathbb{N}} \mathfrak{A}_i$ is the union of a chain of models $\{\mathfrak{A}_{2i+1}\}$ of a $\forall\exists$ theory $T \cup \{\varphi_1\}$, it is also a model of $T \cup \{\varphi_1\}$, by the Chang-Łoś-Suszko theorem.
- Similarly, since $\bigcup_{i\in\mathbb{N}} \mathfrak{A}_i$ is the union of a chain of models $\{\mathfrak{A}_{2i+2}\}$ of a $\forall\exists$ theory $T \cup \{\varphi_2\}$, it is also a model of $T \cup \{\varphi_2\}$.
- Therefore, $\bigcup_{i\in\mathbb{N}} \mathfrak{A}_i$ is a model of $T \cup \{\varphi_1, \varphi_2\}$. So, any model $\mathfrak{A}$ of T can be extended to $T \cup \{\varphi_1, \varphi_2\}$.

Definition 2.8

A theory T is said to be **model complete** if for any model $\mathfrak{A}$, $\mathfrak{B}$ of T,

$$\mathfrak{A} \subseteq \mathfrak{B} \Rightarrow \mathfrak{A} \prec \mathfrak{B}.$$

Lemma 2.9

A model-complete theory is a ∀∃-theory.

Proof. In a model-complete theory T, a chain of models is an elementary chain, so by the elementary chain theorem, the union is also a model of T. By the Chang-Łoś-Suszko theorem, this theory is a ∀∃ theory.

Problem 7. **Homework # 3**

In a model-complete theory, show that for every formula, there exists an equivalent ∀ formula. (Hint. See the proof of (1)⇒(2) in the Łoś-Tarski theorem.)

§3. Horn formula and reduced product

As before, we fix a language $\mathcal{L}$.

Definition 3.1

- $\theta_0 \vee \neg\theta_1 \vee \cdots \vee \neg\theta_n$ or $\neg\theta_1 \vee \cdots \vee \neg\theta_n$ is called a **basic Horn formula**, where $\theta_i\ (i < n)$ are atomic formulas.
- A formula constructed from basic Horn formulas by using only $\wedge$, $\forall$, and $\exists$ is called **Horn formula**.
- The set of Horn sentences is called **Horn theory**.

The basic Horn formula can be expressed as follows, which is easier to use in applications:

$$\theta_1 \wedge \cdots \wedge \theta_n \to \theta_0$$

or

$$\theta_1 \wedge \cdots \wedge \theta_n \to \bot$$

Example 3.

The theory of regular rings, which adds the axiom $\forall x \forall y(xyx = x)$ to ring theory, is a Horn theory.

Theory of integral domain (commutative ring theory + $\forall x \forall y(x \bullet y = 0 \to x = 0 \lor y = 0)$) and field theory (commutative ring theory + $\forall x \exists y(x \neq 0 \to xy = 1)$) is not Horn theory.

The model of Horn's theory is closed under the so-called "reduced products", which is a generalization of direct product.

Before introducing reduced products, we starts with some definitions.

Definition 3.2

Let I be a non-empty set. $\mathcal{F} \subseteq \mathcal{P}(I)$ is called a **filter** on I if the following are satisfied.

(1) $\varnothing \notin \mathcal{F}$, $I \in \mathcal{F}$.

(2) $X \in \mathcal{F},\ X \subseteq Y \subseteq I \Rightarrow Y \in \mathcal{F}$.

(3) $X, Y \in \mathcal{F} \rightarrow X \cap Y \in \mathcal{F}$.

Problem 8

Let I be an infinite set. Show the following.

1. The class of all finite subsets of I is not a filter.
2. The class of all infinite subsets of I is not a filter.
3. The class of subsets X of I such that $I - X$ is finite is a filter, which is called a **Fréchet filter**.
4. For each $i \in I$, $\{X \subseteq I : i \in X\}$ is a filter, which is called a **principal filter**.

Lemma 3.3

If $S \subset \mathcal{P}(I)$ has the **finite intersection property**, i.e., for any finite $\{J_1, \ldots J_n\} \subset S$,

$$J_1 \cap \cdots \cap J_n \neq \varnothing,$$

then there exists a filter $\mathcal{F}$ including S.

Proof. Put $\mathcal{F} = \{X \subseteq I : J_1 \cap \cdots \cap J_n \subset X \text{ for some } \{J_1, \ldots, J_n\} \subset S\}$. □

Definition 3.4

Let $\mathfrak{A}_i = (A_i, \mathrm{f}^{\mathfrak{A}_i}, \ldots, \mathrm{R}^{\mathfrak{A}_i}, \ldots)$ $(i \in I)$ be an $\mathcal{L}$-structure. Let $\mathcal{F}$ be a filter for I. Then, we define the binary relation $\approx_{\mathcal{F}}$ on $\prod A_i$ as follows

$$a \approx_{\mathcal{F}} b \Leftrightarrow \{i \in I : a(i) = b(i)\} \in \mathcal{F}.$$

Lemma 3.5

$\approx_{\mathcal{F}}$ is an equivalence relation.

Proof

- The laws of reflection and symmetricity are clear from the definition.
- To show the transitive law, we assume $a \approx_{\mathcal{F}} b$, $b \approx_{\mathcal{F}} c$.
- By definition,

$$\{i \in I : a(i) = b(i)\} \in \mathcal{F} \text{ and } \{i \in I : b(i) = c(i)\} \in \mathcal{F}.$$

- On the other hand,

$$\{i \in I : a(i) = c(i)\} \supseteq \{i \in I : a(i) = b(i)\} \cap \{i \in I : b(i) = c(i)\}$$

- By conditions (2) and (3) of Definition 3.2, we have

$$\{i \in I : a(i) = c(i)\} \in \mathcal{F}.$$

Therefore, $a \approx_{\mathcal{F}} c$. □

For $a_1, \ldots, a_n \in \prod A_i$, we set $\|\varphi(a_1, \ldots, a_n)\| := \{i \in I : \mathfrak{A}_i \models \varphi(a_1(i), \ldots, a_n(i))\}$.
Note that we write $\mathfrak{A}$ for $\mathfrak{A}_A$ if it is clear from the context.

Lemma 3.6

If $a_1 {\approx_{\mathcal{F}}} b_1, \ldots, a_n {\approx_{\mathcal{F}}} b_n$, we have

$$\|\mathtt{f}(a_1, \ldots, a_n) = \mathtt{f}(b_1, \ldots, b_n)\| \in \mathcal{F},$$

$$\|\mathrm{R}(a_1, \ldots, a_n)\| \in \mathcal{F} \Leftrightarrow \|\mathrm{R}(b_1, \ldots, b_n)\| \in \mathcal{F}.$$

Proof.
This can be derived from the following with conditions (2) and (3) of Definition 3.2.

$$\bigcap_{k \leq n} \{i \in I : a_k(i) = b_k(i)\} \subseteq \|\mathtt{f}(a_1, \ldots, a_n) = \mathtt{f}(b_1, \ldots, b_n)\|,$$

$$\bigcap_{k \leq n} \{i \in I : a_k(i) = b_k(i)\} \cap \|\mathrm{R}(a_1, \ldots, a_n)\| \subseteq \ |\mathrm{R}(b_1, \ldots, b_n)\|.$$

□

Therefore, $\approx_{\mathcal{F}}$ is a congruence relation on $\prod A_i$.

- We can define the quotient structure in the same way as for the algebraic structure.
- That is, its domain is the set of equivalence classes denoted $\prod A_i/{\approx_{\mathcal{F}}}$ or $\prod A_i/\mathcal{F}$, and the value of a function f and the truth value of a relation R is uniquely determined on the equivalence classes regardless of choice of representative elements.

Definition 3.7

Let $\mathfrak{A}_i = (A_i, \mathtt{f}^{\mathfrak{A}_i}, \dots, \mathrm{R}^{\mathfrak{A}_i}, \dots)\ (i \in I)$ be an $\mathcal{L}$-structure. Let $\mathcal{F}$ be a filter for I. Then, $\mathcal{L}$-structure obtained by the Cartesian product $\prod A_i$ dividing by the congruence relation $\approx_{\mathcal{F}}$ is defined by

$$\left(\prod A_i/\mathcal{F}, \mathtt{f}^{\prod \mathfrak{A}_i/\mathcal{F}}, \dots, \mathrm{R}^{\prod \mathfrak{A}_i/\mathcal{F}}, \dots\right)$$

is called the **reduced product** of $\mathfrak{A}_i$'s over $\approx_{\mathcal{F}}$, and is denoted by $\prod \mathfrak{A}_i/\mathcal{F}$.

- For a non-empty set I, $\mathcal{F} = \{I\}$ is a filter and $\prod \mathfrak{A}_i/\mathcal{F} \cong \prod \mathfrak{A}_i$.
  In other words, the direct product is also one kind of the reduced products.
- For a principal filter $\mathcal{F} = \{X \subseteq I : k \in X\}$, $\prod \mathfrak{A}_i/\mathcal{F} \cong \mathfrak{A}_k$.

Lemma 3.8

If φ is a formula obtained from the atomic formula with $\wedge$ and $\exists$, then for $a_1, \ldots, a_n \in \prod A_i$,

$$\prod \mathfrak{A}_i/\mathcal{F} \models \varphi([a_1], \ldots, [a_n]) \Leftrightarrow \|\varphi(a_1, \ldots, a_n)\| \in \mathcal{F}.$$

Proof. By induction on the construction of formulas.

- If φ is an atomic formula, it is clear by the definition.
- If $\varphi = \psi_1 \wedge \psi_2$, it follows from the induction hypo. and the closedness of filter under $\cap$.
- Let $\varphi = \exists x \psi(x)$. For simplicity, we do not display parameters $a_1, \ldots, a_n$ in φ.

$$\begin{aligned}\prod \mathfrak{A}_i/\mathcal{F} \models \exists x \psi(x) &\Leftrightarrow \text{ for some } a \in \prod \mathfrak{A}_i \text{ , } \prod \mathfrak{A}_i/\mathcal{F} \models \psi([a]) \\ &\Leftrightarrow \text{ for some } a \in \prod \mathfrak{A}_i \text{ , } \|\psi(a)\| \in \mathcal{F} \quad \text{(induction hypothesis)} \\ &\Rightarrow \|\exists x \psi(x)\| \in \mathcal{F} \quad (\because \|\psi(a)\| \subseteq \|\exists x \psi(x)\|).\end{aligned}$$

- Conversely, let $\|\exists x \psi(x)\| \in \mathcal{F}$. By the axiom of choice, we take $a \in \prod A_i$ such that for each $i \in \|\exists x \psi(x)\|$, $\mathfrak{A}_i \models \psi(a(i))$. Then, $\|\psi(a)\| \in \mathcal{F}$. By the induction hypothesis, $\prod \mathfrak{A}_i/\mathcal{F} \models \psi([a])$. Therefore, $\prod \mathfrak{A}_i/\mathcal{F} \models \exists x \psi(x)$. □

Lemma 3.9

Let $\varphi(x_1, \ldots, x_n)$ be a basic Horn formula, then for $a_1, \ldots, a_n \in \prod A_i$, we have

$$\|\varphi(a_1, \ldots, a_n)\| \in \mathcal{F} \Rightarrow \prod \mathfrak{A}_i/\mathcal{F} \models \varphi([a_1], \ldots, [a_n]).$$

Proof.

- For simplicity, we do not display parameters $a_1, \ldots, a_n \in \prod A_i$ in the formula.
- Let φ be a basic horn sentence $(\theta_0 \vee)\neg\theta_1 \vee \cdots \vee \neg\theta_n$, where θ_i $(i < n)$ are atomic sentences. We show a contradiction by assuming $\textcircled{1}\|\varphi\| \in \mathcal{F}$ and $\textcircled{2}\prod \mathfrak{A}_i/\mathcal{F} \not\models \varphi$.
- By ②, since $\prod \mathfrak{A}_i/\mathcal{F} \models \theta_1 \wedge \cdots \wedge \theta_n$, by the last lemma, we have $\|\theta_1 \wedge \cdots \wedge \theta_n\| \in \mathcal{F}$.
  - If φ does not contain θ_0, we have $\varnothing = \|\varphi\| \cap \|\theta_1 \wedge \cdots \wedge \theta_n\| \in \mathcal{F}$, which violates the condition of a filter.
  - If φ contains θ_0, we have $\|\theta_0\| = \|\varphi\| \cap \|\theta_1 \wedge \cdots \wedge \theta_n\| \in \mathcal{F}$. Thus by the last lemma, we have $\prod \mathfrak{A}_i/\mathcal{F} \models \theta_0$, which conflicts with the assumption $\prod \mathfrak{A}_i/\mathcal{F} \not\models \varphi$ □

Lemma 3.10

Let $\varphi(x_1, \ldots, x_n)$ be a Horn formula, then for $a_1, \ldots, a_n \in \prod A_i$,

$$\|\varphi(a_1, \ldots, a_n)\| \in \mathcal{F} \Rightarrow \prod \mathfrak{A}_i/\mathcal{F} \models \varphi([a_1], \ldots, [a_n]).$$

Proof. By induction on the construction of a Horn formula with $\wedge$, $\forall$, and $\exists$.
For the basic Horn formula, it follows from the last lemma. Formulas $\varphi \wedge \psi$ and $\exists x \varphi(x)$ are treated in the lemma in Page 35.
For a formula $\forall x \varphi(x)$,

$$\begin{aligned}
\|\forall x \varphi(x)\| \in \mathcal{F} &\Rightarrow \text{for all } a \in \prod A_i,\ \|\varphi(a)\| \in \mathcal{F} \\
&\Rightarrow \text{for all } a \in \prod A_i,\ \prod \mathfrak{A}_i/\mathcal{F} \models \varphi([a]) \\
&\Leftrightarrow \prod \mathfrak{A}_i/\mathcal{F} \models \forall x \varphi(x) \quad \square
\end{aligned}$$

The above lemma shows that the Horn formula preserves reduced products, that is, a reduced product of models of a Horn formula becomes a model of the formula again. Therefore, the class of models of a Horn theory is closed under reduced products, especially under direct products. Then we will show the converse is also true in the next lecture.

Thank you for your attention!