Topics in Applied Math: Logic and Foundations of Mathematics

Part 2. First order theory

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October 10, 2025

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Logic and Foundations

- **Part 1. Equational theory**
- **Part 2. First order theory**
- **Part 3. Basic Model theory**
- **Part 4. First order arithmetic and incompleteness theorems**
- **Part 5. Models of first-order arithmetic**
- **Part 6. Second order arithmetic and reverse mathematics**

Part 2. Schedule

- Oct. 10, (1) First order logic: formal systems and structures
- Oct. 11, (2) Gödel's completeness theorem and applications

Introduction to part 2

- In this part, we will introduce the basics of **first-order logic** (predicate logic) and investigate properties of mathematical theories in this logic.

- Equational theories in the previous part only dealt with equations, but first-order theories can treat more general expressions involving relational symbols (e.g. $<$) and logical symbols (e.g. $\wedge$, $\forall$). Most of ordinary mathematical theories can be formalized in first-order logic.

- We will introduce Gentzen-Tait's deductive system GT, and proves the **Completeness Theorem** (that all valid statements are provable) by Henkin's method.

- By this method, the compactness theorem and Löwenheim-Skolem theorem can also be obtained.

- As an application of completeness theorem, we also present the iterpretation of a theory in another theory.

In part 1, we presented group theory as an equational theory in the language $\mathcal{L} = (\bullet, \mathsf{e}, {}^{-1})$.

Definition 0.1 (in Part 1)

Group theory $\mathrm{G_p}$ consists of the following three axioms.

$$\begin{array}{llcll} \text{G1}: & (x \cdot y) \cdot z & = & x \cdot (y \cdot z) & \text{(associativity)} \\ \text{G2}: & \mathsf{e} \cdot x & = & x & \text{(left identity)} \\ \text{G3}: & x^{-1} \cdot x & = & \mathsf{e} & \text{(left inverse)} \end{array}$$

where x, y and z are variables, e is a constant, and ${}^{-1}$ represents a unary function.

Using first-order logic, it may be formalized without the symbols e, ${}^{-1}$.

Example 1

Group theory can be axiomatized in the language $\mathcal{L} = (\bullet)$ as follows:

$$\begin{cases} (x \cdot y) \cdot z = x \cdot (y \cdot z), \\ \exists z(\forall x(z \cdot x = x) \wedge \forall x \exists y(y \cdot x = z)). \end{cases}$$

Example 2

Various kinds of continuity of a real functions are expressed as follows.

- $f(x)$ is continuous at $x = a$
  $\iff \forall \varepsilon > 0\ \exists \delta > 0\ \forall x(|x - a| < \delta \to |f(x) - f(a)| < \varepsilon)$.
- $f(x)$ is continuous (at all points)
  $\iff \forall a \forall \varepsilon > 0\ \exists \delta > 0\ \forall x(|x - a| < \delta \to |f(x) - f(a)| < \varepsilon)$.
- $f(x)$ is uniformly continuous
  $\iff \forall \varepsilon > 0\ \exists \delta > 0\ \forall x \forall y(|x - y| < \delta \to |f(x) - f(y)| < \varepsilon)$.

Note that $\forall x > 0$ is an abbreviation for $\forall x(x > 0 \to \cdots)$, $\exists x > 0$ is for $\exists x(x > 0 \wedge \cdots)$.

What is first order logic?

- Propositional logic is the study of logical connections between propositions expressed by $\neg, \wedge, \vee, \rightarrow$.
- First-order logic is obtained from propositional logic by adding logical symbols $\forall, \exists$.

  $\forall x$ expresses "for every element x (of the underlying set)", and
  $\exists x$ expresses "there exists an element x (of the underlying set)".

  $\forall x$ is called a **universal quantifier**, and $\exists x$ is called an **existencial quantifier**.
- Historically, first-order logic was introduced by D. Hilbert as a downsized system of Russell's type theory to handle mathematical theories in more algebraic way. He then asked whether his formulation is complete (i.e., sufficient to prove all the valid formulas). Gödel answered the question affirmatively in his doctoral thesis.
- Hilbert also proposed the decision problem (such as satisfiability) of first-order logic as "**the main problem of mathematical logic** (Hauptproblem)" (1928). Then, Gödel showed the undecidability of first-order arithmetic. Subsequently, Church and Turing proved that first-order logic is undecidable.

§1. Languages and structures

Definition 1.1

A **language** (**signature, alphabet**) $\mathcal{L}$ of first-order logic is a list or set of function symbols $\mathtt{f}_i$ and relational symbols $\mathtt{R}_j$ denoted as

$$\mathcal{L} = (\mathtt{f}_0, \mathtt{f}_1, \ldots; \mathtt{R}_0, \mathtt{R}_1, \ldots).$$

If $\mathtt{f}_i$ is an m_i-ary function symbol (for each i) and $\mathtt{R}_j$ is an n_j-ary relation symbol (for each j), then $\rho = (m_0, m_1, \ldots; n_0, n_1, \ldots)$ is called a **similarity type** of language $\mathcal{L}$.

Note that $\mathcal{L}$ may be uncountable.

Function symbols $\mathrm{f}, \cdots$ and **relation symbols** $\mathrm{R}, \cdots$ are mathematical symbols of a specific theory, in addition to the following common logical symbols:

1. **propositional connectives**: $\neg$ (not $\cdots$), $\land$ (and), $\lor$ (or), $\to$ (implies),
2. **quantifiers**: $\forall$ (for all $\cdots$), $\exists$ (there exists $\cdots$).
3. **variables**: $x_0, x_1, \cdots$
4. **equality** $=$ and auxiliary symbols such as parentheses $($, $)$.

Definition 1.2

A **structure** in language $\mathcal{L}$ (an $\mathcal{L}$-structure) is defined as a non-empty set A equipped with an interpretation of the symbols in $\mathcal{L}$, denoted as

$$\mathfrak{A} = (A;\ \mathrm{f}^{\mathfrak{A}}, \cdots, \mathrm{R}^{\mathfrak{A}}, \cdots).$$

- $A = |\mathfrak{A}|$ is called the **domain** (or universe) of the structure $\mathfrak{A}$, where A is a non-empty.
- For an m-ary function symbol $\mathrm{f} \in \mathcal{L}$, $\mathrm{f}^{\mathfrak{A}} : A^m \to A$.
- For an n-ary relational symbol $\mathrm{R} \in \mathcal{L}$, $\mathrm{R}^{\mathfrak{A}} \subseteq A^n$.

A function symbol with no argument (0-ary function) is called a **constant**.
Since a constant plays a special role distinct from a function, they are often treated separately. In such a case, a language $\mathcal{L}$ may be flexibly written as $(\mathrm{c}, \ldots; \mathtt{f}, \ldots; \mathtt{R}, \ldots)$, and an $\mathcal{L}$-structure as $(A;\ \mathrm{c}^{\mathfrak{A}}, \cdots, \mathrm{f}^{\mathfrak{A}}, \cdots, \mathrm{R}^{\mathfrak{A}}, \cdots)$, where $\mathrm{c}^{\mathfrak{A}} \in A$.

Similarly to the algebraic language (Part 1, §1), a **term** of first-order language $\mathcal{L}$ is a symbol string consisting of variables and function symbols of $\mathcal{L}$.
A term that includes no variables is called a **closed term**, which indicates an element of the structure determined by the following definition.

Definition 1.3 (Terms)

The **terms** of the language $\mathcal{L}$ are symbol strings defined inductively as follows.

1. variables (and constants in $\mathcal{L}$) are terms of $\mathcal{L}$.
2. If $t_0, \cdots, t_{n-1}$ are terms and f is an n-ary function symbol of $\mathcal{L}$, then $\mathrm{f}(t_0, \cdots, t_{n-1})$ is a term of $\mathcal{L}$.

A term t with no variables is called a **closed** term. The **value** of a close term t in an $\mathcal{L}$-structure $\mathfrak{A}$, denoted $t^{\mathfrak{A}}$, is defined inductively as follows.

1. the value of constant c in $\mathcal{L}$ is $\mathrm{c}^{\mathfrak{A}}$.
2. the value of term $\mathrm{f}(t_0, \cdots, t_{n-1})$ is $\mathrm{f}^{\mathfrak{A}}(t_1^{\mathfrak{A}}, \cdots, t_{n-1}^{\mathfrak{A}})$.

Definition 1.4 (Formulas)

A **formula** of language $\mathcal{L}$ is a sequence of symbols inductively defined as follows.

(1) If $s, t, t_0, \cdots, t_{n-1}$ are terms of $\mathcal{L}$, and R is an n-ary relation symbol of $\mathcal{L}$, then

$$s = t \quad \text{and} \quad \mathrm{R}(t_0, \cdots, t_{n-1})$$

are formulas of $\mathcal{L}$, which are called **atomic** formulas.

(2) If φ, ψ are formulas of $\mathcal{L}$, then so are the followings: for any variable x,

$$\neg(\varphi),\ (\varphi) \land (\psi),\ (\varphi) \lor (\psi),\ (\varphi) \to (\psi),\ \ \forall x(\varphi),\ \ \exists x(\varphi).$$

The brackets (,) can be omitted if no confusion might occur.

Example 3

- Let $\mathcal{L}_{\mathrm{OR}} = (+, \bullet, 0, 1; <)$ be the language of ordered rings, whose similarity type is $\rho = (2, 2, 0, 0; 2)$.
- The standard structure $\mathfrak{N} = (\mathbb{N}; +, \bullet, 0, 1; <)$ of natural numbers is an $\mathcal{L}_{\mathrm{OR}}$-structure, where $+, \bullet, 0, 1$ and $<$ represent the ordinary functions and relation on natural numbers. For example, 1 in the structure is not a constant symbol, but an element of $\mathbb{N}$ indicated by constant 1.
- In this language,
  (1) $(x_0 + 1) \bullet x_2$ is a term.
  (2) $(x_0 + 1) \bullet x_2 < x_1$ is an atomic formula.
  (3) $\forall x_0((x_0 + 1) \bullet x_2 < x_1) \wedge \forall x_1 \exists x_3(x_1 \bullet x_2 = x_3)$ is a formula.

A formula that appears in the process of constructing a formula is called its **subformulas**. The subformulas of formula (3) in the above example are

- the two atomic subformulas $(x_0 + 1) \bullet x_2 < x_1$, $x_1 \bullet x_2 = x_3$,
- $\forall x_0((x_0 + 1) \bullet x_2 < x_1)$, $\exists x_3(x_1 \bullet x_2 = x_3)$, $\forall x_1 \exists x_3(x_1 \bullet x_2 = x_3)$, and the whole formula.

- In a formula, a variable x is said to be **bound** if it occurs in a subformula of the form $\forall x\varphi(x)$ or $\exists x\varphi(x)$, and otherwise it is said to be **free**.
- In (3) $\forall x_0((x_0+1)\bullet x_2 < x_1) \land \forall x_1 \exists x_3(x_1 \bullet x_2 = x_3)$ of Example 3,
  x_0 and x_3 are bound, x_2 is free, x_1 appears both free and bound.
- Focusing on the free occurrence of some variables (say, x, y) in a formula φ, we write it as $\varphi(x, y)$.
- Then, we write $\varphi(s)$ for the formula obtained from $\varphi(x)$ by substituting a term s into every free occurrence of x.
- If a variable y included in s will be bound in $\varphi(s)$, we replace the bound variable y in φ with a new variable z in advance.
- For example, consider the substitution of $x = s$ for $\forall y\varphi(x, y)$, where s includes y. In such a case, change $\forall y\varphi(x, y)$ with $\forall z\varphi(x, z)$ for the first. Then by substituting $x = s$ for $\forall z\varphi(x, z)$, we obtain $\forall z\varphi(s, z)$. Such a replacement of variables are automatically done to avoid confusion.

Definition 1.5

- A formula φ with no free variables is called a sentence or a **closed formula**.
- A formula φ with no quantifiers is called a **quantifier-free** formula or **open formula**. Obviously, any variable in an open formula is free.

- For a formula $\varphi(x_1, \cdots, x_n)$ with no free variables unspecified, a sentence of the form $\forall x_1 \forall x_2 \ldots \forall x_n \varphi$ is called the **universal closure** of φ.
- A formula and its universal closure are often identified. For example, $x < x + 1$ is interchangeable with $\forall x(x < x + 1)$.
- Strictly, a **theory** is a set of sentences. But we may also treat a set of formulas as a theory by replacing each formula with its universal closure.

Our next goal is to define the concept of truth/falsity of sentences. First, the truth/falsity of an atomic sentence (an atomic formula with no variables) is defined as follows.

Definition 1.6

Let $\mathfrak{A}$ be a structure in language $\mathcal{L}$.

(1) Let s and t be closed terms. If $s^{\mathfrak{A}}$ and $t^{\mathfrak{A}}$ have the same value,
the atomic formula $s = t$ is **true** in $\mathfrak{A}$; otherwise **false**.

(2) Let $\mathtt{R}$ be an n-ary relation symbol in $\mathcal{L}$, and $t_0, \ldots, t_{n-1}$ be closed terms.
If $\mathtt{R}^{\mathfrak{A}}(t_0^{\mathfrak{A}}, \ldots, t_{n-1}^{\mathfrak{A}})$ holds, then
the atomic formula $\mathtt{R}(t_0, \ldots, t_{n-1})$ is **true** in $\mathfrak{A}$; otherwise **false**.

In order to argue the true/false of general sentences, we first introduce an expansion and a reduct of a structure $\mathfrak{A}$.

Definition 1.7

Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures in languages $\mathcal{L}$ and $\mathcal{L}'$, respectively, and $\mathcal{L} \subset \mathcal{L}'$.
Moreover, suppose that $|\mathfrak{A}| = |\mathfrak{B}|$ and for each symbol $\mathtt{f}$, $\mathtt{R}$ of $\mathcal{L}$, $\mathtt{f}^{\mathfrak{A}}(\vec{x}) = \mathtt{f}^{\mathfrak{B}}(\vec{x})$ and $\mathtt{R}^{\mathfrak{A}}(\vec{x}) \Leftrightarrow \mathtt{R}^{\mathfrak{B}}(\vec{x})$.
Then, $\mathfrak{B}$ is called an **expansion** of $\mathfrak{A}$ or $\mathfrak{A}$ is called an **reduct** of $\mathfrak{B}$.

Let C be a subset of the domain $A = |\mathfrak{A}|$, and c^* denote a new constant for each element $c \in C$. Then, put $\mathcal{L}_C = \mathcal{L} \cup \{c^* : c \in C\}$. Now, the structure $\mathfrak{A}$ is expanded to a $\mathcal{L}_C$-structure $\mathfrak{A}_C$ with constant c^* interpreted as c. In particular, $\mathfrak{A}_\varnothing = \mathfrak{A}$.

The concept of true/false of a sentence in a structure $\mathfrak{A}$ is defined via the extended structure $\mathfrak{A}_A$. For simplicity, we do not distinguish an element a of $|\mathfrak{A}|$ and a constant a^*.

Definition 1.8 (Tarski's truth definition clauses)

The set of true sentences in the structure $\mathfrak{A}_A$, denoted $\mathrm{Th}(\mathfrak{A}_\mathrm{A})$, is defined inductively by the following **Tarski's truth definition clauses**.

- For an atomic sentence φ of $\mathcal{L}_A$, $\varphi \in \mathrm{Th}(\mathfrak{A}_\mathrm{A}) \Leftrightarrow \varphi$ is true in $\mathfrak{A}_A$,
- $\neg\varphi \in \mathrm{Th}(\mathfrak{A}_\mathrm{A}) \Leftrightarrow \varphi \notin \mathrm{Th}(\mathfrak{A}_\mathrm{A})$,
- $\varphi \land \psi \in \mathrm{Th}(\mathfrak{A}_\mathrm{A}) \Leftrightarrow \varphi \in \mathrm{Th}(\mathfrak{A}_\mathrm{A})$ and $\psi \in \mathrm{Th}(\mathfrak{A}_\mathrm{A})$,
- $\varphi \lor \psi \in \mathrm{Th}(\mathfrak{A}_\mathrm{A}) \Leftrightarrow \varphi \in \mathrm{Th}(\mathfrak{A}_\mathrm{A})$ or $\psi \in \mathrm{Th}(\mathfrak{A}_\mathrm{A})$,
- $\varphi \to \psi \in \mathrm{Th}(\mathfrak{A}_\mathrm{A}) \Leftrightarrow \varphi \notin \mathrm{Th}(\mathfrak{A}_\mathrm{A})$ or $\psi \in \mathrm{Th}(\mathfrak{A}_\mathrm{A})$,
- $\forall x\varphi(x) \in \mathrm{Th}(\mathfrak{A}_\mathrm{A}) \Leftrightarrow$ for every $a \in A$, $\varphi(a) \in \mathrm{Th}(\mathfrak{A}_\mathrm{A})$,
- $\exists x\varphi(x) \in \mathrm{Th}(\mathfrak{A}_\mathrm{A}) \Leftrightarrow$ there exists $a \in A$ such that $\varphi(a) \in \mathrm{Th}(\mathfrak{A}_\mathrm{A})$.

$\mathrm{Th}(\mathfrak{A}_\mathrm{A})$ is called the **elementary diagram** of the structure $\mathfrak{A}$. The set of atomic sentences and negations of atomic sentences included in $\mathrm{Th}(\mathfrak{A}_\mathrm{A})$ is called the **(basic) diagram**, which is denoted as $\mathrm{Diag}(\mathfrak{A})$.

Definition 1.9

If an $\mathcal{L}$-sentence φ belongs to $\mathrm{Th}(\mathfrak{A}_{\mathrm{A}})$, we say φ is **true** in the structure $\mathfrak{A}$, written as $\mathfrak{A} \models \varphi$ or $\varphi \in \mathrm{Th}(\mathfrak{A})$.
For an $\mathcal{L}$-formula φ, if its universal closure $\forall x_0 \cdots \forall x_{n-1}\varphi$ is true in $\mathfrak{A}$, we write $\mathfrak{A} \models \varphi$.

Definition 1.10

If a set T of sentences in language $\mathcal{L}$ is a subset of $\mathrm{Th}(\mathfrak{A})$, the structure $\mathfrak{A}$ is called a **model** of T, written as $\mathfrak{A} \models T$.

Definition 1.11

Let T be a set of sentences in language $\mathcal{L}$, and φ a formula in $\mathcal{L}$. If $\mathfrak{A} \models \varphi$ for any model $\mathfrak{A}$ of T, we say that φ is a **consequence** of T, written as $T \models \varphi$.

$T \models \varphi$ represents that a sentence φ holds in a theory T in the ordinary mathematical sense. This will be formalized in a formal deductive system in the next section.

§2. Gentzen-Tait formal system GT

- A logical deduction system consists of axioms and inference rules.
  A deductive system mainly based on axioms is called the **Hilbert style**,
  and contrarily one mainly based on inference rules is called the **Gentzen style**.
- In the following, we adopt a Gentzen-style system modified by Tait, called **GT**.
- In GT, treat $\varphi \to \psi$ as an abbreviation for $\neg\varphi \vee \psi$. Then, the key feature of GT is that any formula φ is automatically transformed into the negation normal form, i.e., constructed from atomic formulas or their negations by means of $\wedge$, $\vee$, $\forall$, and $\exists$.
  In the following, $\neg s = t$ is also written as $s \neq t$.
- To keep the negation normal form, the following replacement rules (**De Morgan's laws**) are applied wherever possible:

$$\neg(\varphi \vee \psi) := \neg\varphi \wedge \neg\psi, \qquad \neg(\varphi \wedge \psi) := \neg\varphi \vee \neg\psi,$$

$$\neg\forall x\varphi := \exists x\neg\varphi, \qquad \neg\exists x\varphi := \forall x\neg\varphi, \qquad \neg\neg\varphi := \varphi.$$

Definition 2.1

- A finite sequence of formulas $\varphi_1, \ldots, \varphi_n$ is called a **sequent**.
- All sequents formed by rearranging the elements are regarded as the same.
- A sequent is therefore a multiset rather than a sequence.
- For two sequents Γ $(= \varphi_1, \ldots, \varphi_n)$ and Δ $(= \psi_1, \ldots, \psi_m)$, Γ, Δ denotes the sequent $\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_m$.

The sequent $\varphi_1, \ldots, \varphi_n$ intuitively means $\varphi_1 \vee \cdots \vee \varphi_n$. The Gentzen-Tait system is a deductive system of sequents.

Definition 2.2 (Gentzen-Tait system)

Gentzen-Tait system $\mathrm{GT}(T)$ of a theory T has the following axioms and inference rules:

Axioms

(0) φ (where $\varphi \in T$)

(1) Law of excluded middle: $\neg\psi, \psi$ (where ψ is an atomic formula)

(2) Equality axioms:

- (i) $x = x$,
- (ii) $x \neq y, y = x$,
- (iii) $x \neq y, y \neq z, x = z$,
- (iv) $x_1 \neq y_1, \ldots, x_m \neq y_m, \mathtt{f}(x_1, \ldots, x_m) = \mathtt{f}(y_1, \ldots, y_m)$,
- (v) $x_1 \neq y_1, \ldots, x_n \neq y_n, \mathtt{R}(x_1, \ldots, x_n), \neg\mathtt{R}(y_1, \ldots, y_n)$,

Definition 2.2 (Gentzen-Tait system)

Inference rules

$$\frac{\Gamma, \varphi, \psi}{\Gamma, \varphi \vee \psi}\ (\vee), \quad \frac{\Gamma, \varphi \quad \Gamma, \psi}{\Gamma, \varphi \wedge \psi}\ (\wedge)$$

$$\frac{\Gamma, \varphi(t)}{\Gamma, \exists x \varphi(x)}\ (\exists), \quad \frac{\Gamma, \varphi(x)}{\Gamma, \forall x \varphi(x)}\ (\forall)$$ where Γ has no free occurrences of x

$$\frac{\Gamma}{\Delta}\ (\mathrm{weak}^1)$$ where Γ is a subsequence of Δ, $$\frac{\Gamma, \neg A \quad \Gamma, A}{\Gamma}\ (\mathrm{cut})$$

[1] the weakening rule

Definition 2.3

- A **proof tree** in the system GT(T) is a finite tree in which each vertex is labelled with a sequent so that a sequent at each top vertex (leaf) is an axiom, and the sequents of adjacent nodes express an inference rule. See the examples below.
- If there is a proof tree rooted at the sequent Γ, we write it as $T \vdash \Gamma$. Such a tree is called a **proof** of $T \vdash \Gamma$ (or a **proof** of Γ in T).
- If $T = \varnothing$ or T is clear from the context, we omit T and write $\vdash \Gamma$.

Example 4

A proof of $\vdash \neg\varphi \vee (\neg\psi \vee (\varphi \wedge \psi))$, where φ, ψ are atomic formulas, is following.

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg\varphi, \varphi}{\neg\varphi, \varphi, \neg\psi}\,(\text{weak}) \qquad \dfrac{\neg\psi, \psi}{\neg\psi, \psi, \neg\varphi}\,(\text{weak})}{\neg\varphi, \neg\psi, \varphi \wedge \psi}\,(\wedge)}{\neg\varphi, \neg\psi \vee (\varphi \wedge \psi)}\,(\vee)}{\neg\varphi \vee (\neg\psi \vee (\varphi \wedge \psi))}\,(\vee)$$

Example 5

For any term t, we have $\vdash t = t$ as follows.

$$\dfrac{\dfrac{\dfrac{x = x}{\forall x(x = x)}\ (\forall)}{\forall x(x = x), t = t}\ (\text{weak}) \qquad \dfrac{t \neq t, t = t}{\exists x(x \neq x), t = t}\ (\exists)}{t = t}\ (\text{cut})$$

Here, note that $\neg\exists x(x \neq x)$ can be rewritten as $\forall x(x = x)$. Any substitution instance of other equality axioms can also be proved in the same way as Example 5.

Problem

Consider the case that $\mathtt{R}$ is the equality in the equational axiom

$$x_1 \neq y_1, \ldots, x_n \neq y_n, \mathtt{R}(x_1, \ldots, x_n), \neg\mathtt{R}(y_1, \ldots, y_n).$$

Show that the equational axioms (ii) and (iii) can be derived from the above axiom and the equational axiom (i) in Page 20.

Lemma 2.4

$\vdash \neg\varphi, \varphi$ for any formula φ.

Proof. By induction on the construction of φ.
If φ is an atomic formula, it is an axiom.
If $\varphi \equiv \psi \vee \theta$ then $\neg\varphi \equiv \neg\psi \wedge \neg\theta$ and

$$\dfrac{\dfrac{\dfrac{\neg\psi, \psi}{\neg\psi, \psi, \theta}\ (\mathrm{weak})}{\neg\psi, \psi \vee \theta}\ (\vee) \qquad \dfrac{\dfrac{\neg\theta, \theta}{\neg\theta, \psi, \theta}\ (\mathrm{weak})}{\neg\theta, \psi \vee \theta}\ (\vee)}{\neg\psi \wedge \neg\theta, \psi \vee \theta}\ (\wedge)$$

Formulas φ of other forms can be proved in the same way (Exercise). □

By this lemma, we can see that in Example 4 in Page 22, φ, ψ may not only be atomic formulas but any formulas.

§3. Gödel's completeness theorem

To prepare for a proof of the completeness theorem for GT, we will show several lemmas.

Lemma 3.1 (Deduction theorem)

Let T be an $\mathcal{L}$-theory, φ a sentence and Γ be a sequent. Then,

$$T \cup \{\varphi\} \vdash \Gamma \Rightarrow T \vdash \neg\varphi, \Gamma.$$

Proof.

- Let P be a proof tree of $T \cup \{\varphi\} \vdash \Gamma$. Then let P' be a tree obtained from P by adding the formula $\neg\varphi$ to all the sequents appearing in the proof tree.
- Any inference rule at adjacent vertices remains as the same kind of inference rule. (Note: Since $\neg\varphi$ is a sentence, the condition of the rule $(\forall)$ holds.)
- If a sequent at a leaf of P is an axiom Δ of T, then the corresponding vertex of P' is labelled $\neg\varphi, \Delta$. So, we add a vertex labelled axiom Δ above it so that they satisfy the inference rule (weak).
- If a sequent at a leaf of P is φ, the corresponding sequent of P' is $\neg\varphi, \varphi$, that is, the law of excluded middle. So we put an appropriate proof of it above it. Then let P'' be a tree obtained from P' by doing all these modifications.
- Thus, P'' is a proof tree of $T \vdash \neg\varphi, \Gamma$. □

Remark on the deduction theorem:

- Be careful of the difference between assuming φ in theory T and assuming $T \vdash \varphi$. Actually, if we assume $T \vdash \varphi$ with $T =$ ZF, $\varphi =$ AC, this leads to a contradiction.

Definition 3.2

T is said to be **inconsistent** if $T \vdash$ (i.e., T proves the empty sequent). Otherwise, T is said to be **consistent**.

Lemma 3.3

Let T be a theory and φ asentence. The following hold.

(1) If there exists a φ such that $T \vdash \varphi$ and $T \vdash \neg\varphi$, then T is inconsistent.

(2) $T \cup \{\neg\varphi\}$ is inconsistent $\Leftrightarrow T \vdash \varphi$.

(3) $T \cup \{\neg\varphi\}$ is consistent $\Leftrightarrow T \vdash \varphi$ does not hold .

Proof.

- (1) can be obtained by using the inference rule (cut).
- ($\Rightarrow$) of (2) is nothing but the deduction theorem. To show ($\Leftarrow$) of (2), assume $T \vdash \varphi$. Then $T \cup \{\neg\varphi\} \vdash \varphi$. Also, $T \cup \{\neg\varphi\} \vdash \neg\varphi$. So by (1), $T \cup \{\neg\varphi\}$ is inconsistent.
- (3) is the contrapositive of (2).

□

Lemma 3.4

If T is consistent, $T \cup \{\varphi\}$ or $T \cup \{\neg\varphi\}$ are consistent for any sentence φ.

Proof. It is clear from (1) and (3) in the above lemma.

Lemma 3.5

Let $T \cup \{\exists x \varphi(x)\}$ be consistent and c be a new constant. Then $T \cup \{\varphi(\mathsf{c})\}$ is also consistent in $\mathcal{L}' = \mathcal{L} \cup \{\mathsf{c}\}$.

Proof.

- By way of contradiction, assume $T \cup \{\varphi(\mathsf{c})\} \vdash$.
- By the deduction theorem, we have $T \vdash \neg\varphi(\mathsf{c})$.
- Now, let x be a variable that does not appear in the proof of $T \vdash \neg\varphi(\mathsf{c})$. If we replace all c with x in the proof, we have a proof of $T \vdash \neg\varphi(x)$.
- If we add an inference rule $(\forall)$ at the root of the proof tree, we have a proof of $T \vdash \forall x \neg\varphi(x)$, that is, $T \vdash \neg\exists x \varphi(x)$.
- Therefore, by (2) of Lemma 3.3 in Page 26, $T \cup \{\exists x \varphi(x)\}$ is inconsistent, which contradicts with our assumption.

Lemma 3.6

Let $\mathcal{L}$ be a language. Then there exists a set C of constants not included in $\mathcal{L}$ and a set H of sentences in $\mathcal{L}' = \mathcal{L} \cup C$ such that for any consistent $\mathcal{L}$-theory T, the following hold:

(1) $T \cup H$ is consistent.

(2) For each $\mathcal{L}'$-sentence $\exists x \varphi(x)$ such that $T \cup H \vdash \exists x \varphi(x)$, there exists $\mathsf{c} \in C$ such that $T \cup H \vdash \varphi(\mathsf{c})$.

$T \cup H$ is called the **Henkin extension** or the **Henkinization** of T.

- The above lemma is the core of the proof of Gödel's completeness theorem for GT.
- In the proof of Birkhoff's completeness theorem, we created a model by dividing the term algebra by the congruence relation.
- The basic idea to construct a model of first-order logic is the same. But the term algebra of the given language is not sufficient to be the universal structure.
- So, we extend the language by introducing many new constants called the **Henkin constants**. This extension depends on a language, not a theory.

Proof. We describe how to construct C and H of the lemma. For each $\mathcal{L}$-sentence $\exists x\varphi(x)$, we add new constant $\mathsf{c}_{\exists x\varphi(x)}$, and collect them as C_1, i.e.,

$$C_1 = \{\mathsf{c}_{\exists x\varphi(x)} : \exists x\varphi(x) \text{ is an } \mathcal{L}\text{-sentence}\}.$$

For each constant $\mathsf{c}_{\exists x\varphi(x)} \in C_1$, we define a sentence of the form $\neg\exists x\varphi(x) \vee \varphi(\mathsf{c}_{\exists x\varphi(x)})$ and we collect them as H_1, i.e.,

$$H_1 = \{\neg\exists x\varphi(x) \vee \varphi(\mathsf{c}_{\exists x\varphi(x)}) : \mathsf{c}_{\exists x\varphi(x)} \in C_1\}$$

From the law of excluded middle $\neg\varphi(x),\ \varphi(x)$, we have

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg\varphi(x), \varphi(x)}{\neg\varphi(x), \neg\exists x\varphi(x), \varphi(x)}\,(\text{weak})}{\neg\varphi(x), \neg\exists x\varphi(x) \vee \varphi(x)}\,(\vee)}{\neg\varphi(x), \exists x(\neg\exists x\varphi(x) \vee \varphi(x))}\,(\exists)}{\forall x\neg\varphi(x), \exists x(\neg\exists x\varphi(x) \vee \varphi(x))}\,(\forall)$$

Also, from $\exists x\varphi(x),\ \neg\exists x\varphi(x)$, we have

$$\exists x\varphi(x), \exists x(\neg\exists x\varphi(x) \vee \varphi(x)).$$

Applying the inference rule (cut), we obtain

$$\vdash \exists x(\neg\exists x\varphi(x) \vee \varphi(x)).$$

Proof (continued). Suppose T is consistent. Then, $T \cup \{\exists x(\neg\exists x\varphi(x) \vee \varphi(x))\}$ is also consistent. Thus by Lemma 3.5 in Page 27, $T \cup H_1$ is consistent.
Similarly, for each sentence of the form $\exists x\varphi(x)$ in $\mathcal{L}_1 = \mathcal{L} \cup C_1$, we add the constant $\mathsf{c}_{\exists x\varphi(x)}$ and collect them as $C_2 \supseteq C_1$; collect the sentences $\neg\exists x\varphi(x) \vee \varphi(\mathsf{c}_{\exists x\varphi(x)})$ as $H_2 \supseteq H_1$:

$$C_2 = \{\mathsf{c}_{\exists x\varphi(x)} : \exists x\varphi(x) \text{ is an } \mathcal{L}_1\text{-}sentence\}.$$

$$H_2 = \{\neg\exists x\varphi(x) \vee \varphi(\mathsf{c}_{\exists x\varphi(x)}) : \mathsf{c}_{\exists x\varphi(x)} \in C_2\}$$

For any consistent theory T, $T \cup H_2$ is consistent. By repeating this process, we construct two increasing sequences

$$C_0 = \varnothing \subseteq C_1 \subseteq C_2 \subseteq \cdots, \qquad H_0 = \varnothing \subseteq H_1 \subseteq H_2 \subseteq \cdots$$

and we set

$$C = \bigcup_{i \in \mathbb{N}} C_i \text{ and } H = \bigcup_{i \in \mathbb{N}} H_i.$$

A constant that belongs to C is called a **Henkin constant**, and a sentence that belongs to H is called a **Henkin axiom**.

Proof (continued). Next we show that C and H constructed above satisfy conditions (1) and (2) of the lemma.
Let T be a consistent $\mathcal{L}$-theory.
(1) If $T \cup H$ were inconsistent, then the inconsistency results from some finite segment of $T \cup H$, that is, there would exist $i \in \mathbb{N}$ such that $T \cup H_i$ were inconsistent. This contradicts the construction of $\{H_i\}$.
(2) Let $\mathcal{L}' = \mathcal{L} \cup C$ and $\exists x \varphi(x)$ be an $\mathcal{L}'$-sentence. Then $\exists x \varphi(x)$ is an $\mathcal{L}_i$-sentence of $\mathcal{L} \cup C_i$ for some $i \in \mathbb{N}$. So

$$\neg \exists x \varphi(x) \vee \varphi(\mathsf{c}_{\exists x \varphi(x)}) \in H_{i+1} \subseteq H.$$

Now, we assume $T \cup H \vdash \exists x \varphi(x)$. By a simple calculation, we also have $T \cup H \vdash \exists x \varphi(x) \wedge \neg \varphi(\mathsf{c}_{\exists x \varphi(x)}), \varphi(\mathsf{c}_{\exists x \varphi(x)})$. Then by inference rule (cut), we have

$$T \cup H \vdash \varphi(\mathsf{c}_{\exists x \varphi(x)}).$$

□

Remark

- The cardinality of C constructed in the above proof coincides with the larger one of the cardinality of $\mathcal{L}$ and the countable infinity. In particular, if $\mathcal{L}$ is finite or countably infinite, C is countably infinite.
- This can be seen from the fact that each C_{i+1} is no larger than the number of formulas in $\mathcal{L} \cup C_i$.
- Note that the set of finite sequences from countably many symbols is countably infinite. The set of finite sequences from uncountable κ symbols has cardinality κ.

Homework # 1

1. Let $\mathcal{L} = (<, f)$. In the structure $(\mathbb{R}, <, f)$ of real numbers with ordinary $<$, construct a formula expressing "the function $f(x)$ is continuous at $x = a$". (Cf. Example 2, p.6. Here, you are not allowed to use operations $+, \cdot, -$.)

2. Let $(\mathbb{R}, <, f)$ be the same $\mathcal{L}$-structure as above. Then, show that there is no formula that expresses "$f(x)$ is differentiable at $x = a$".

Thank you for your attention!