Topics in Applied Math:
Logic and Foundations of Mathematics

Part 1. Equational theory

Kazuyuki Tanaka

BIMSA

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Qiuzhen College, Tsinghua University

Logic and Foundations

- **Part 1. Equational theory**
- **Part 2. First order theory**
- **Part 3. Basic Model theory**
- **Part 4. First order arithmetic and incompleteness theorems**
- **Part 5. Models of first-order arithmetic**
- **Part 6. Second order arithmetic and reverse mathematics**

Part 1. Schedule

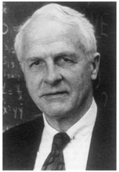
- Sep. 17, (1) Formal systems of equations
- Sep. 19, (2) Birkhoff's theorems and Boolean algebras
- Sep. 24, (3) Boolean algebras (continued) and propositional logic
- Sep. 26, (4) Computable functions and general recursive functions

Recap: Birkhoff's theorems

- For an equational theory T, the following holds.

Birkhoff's completeness theorem (1935)

$$T \models s = t \Leftrightarrow T \vdash s = t.$$



Garrett Birkhoff

- $T \models s = t \Leftarrow T \vdash s = t$ (the soundness of T) is easy. Let $\mathfrak{M}$ be any model of T. Then we can show by induction that all equations appearing in a proof tree for $T \vdash s = t$ holds in $\mathfrak{M}$. Especially the bottom $s = t$ holds in $\mathfrak{M}$.
- To show the contrapositive, we first assume $T \nvdash s = t$, and construct a structure $\mathfrak{M}$ such that $\mathfrak{M} \models T$ and $\mathfrak{M} \not\models s = t$. Such a structure is obtained as the "free algebra" generated by the variables appearing in s, t.

Variety theorem

A class $\mathcal{K}$ of structures is characterized by an equational theory $\Leftrightarrow$ $\mathcal{K}$ is closed under

- subalgebras,
- homomorphisms,
- Cartesian products.

Definition 3.3

Let $\mathcal{K}$ be a class of $\mathcal{L}$-algebras. $\mathfrak{A} \in \mathcal{K}$ is a **free $\mathcal{K}$-algebra** generated by $X \subseteq |\mathfrak{A}|$ if

1. $\mathfrak{A}$ is generated by X, that is, it has no proper subalgebra containing X.
2. Every map $\phi : X \to |\mathfrak{B}|$ with $\mathfrak{B} \in \mathcal{K}$ can be uniquely extended to the homomorphism $\hat{\phi} : \mathfrak{A} \to \mathfrak{B}$.

$$\begin{tikzcd}
X \arrow[r, hook, "\mathrm{id}_X"] \arrow[dr, "\phi"'] & \mathfrak{A} \arrow[d, "\hat{\phi}"] \\
 & \mathcal{B}
\end{tikzcd}$$

An $\mathcal{L}$-algebra (Term(X), $\mathsf{f}_0^{\mathcal{T}(X)}, \mathtt{f}_1^{\mathcal{T}(X)}, \dots$) is a **term algebra**, if $\mathrm{Term}(X)$ is the set of $\mathcal{L}$-terms with variables in X and for each function symbol $\mathtt{f}$ in $\mathcal{L}$,

$$\mathtt{f}^{\mathcal{T}(X)}(t_0, \dots, t_{n-1}) = \mathtt{f}(t_0, \dots, t_{n-1}).$$

Lemma 3.5

If a class of $\mathcal{L}$-algebra $\mathcal{K}$ contains $\mathcal{T}(X)$, then $\mathcal{T}(X)$ is a free $\mathcal{K}$-algebra generated by X.

Definition 3.6

$\mathfrak{A} \models \mathbf{s} = \mathbf{t}$ if for every homomorphism $\phi : \mathcal{T}(X) \to \mathfrak{A}$, we have $\phi(s) = \phi(t)$.

A homomorphism $\phi : \mathcal{T}(X) \to \mathfrak{A}$ can be viewed as an evaluation function of terms. The value of a term s is uniquely obtained from the values $\phi(x)$ for variables x in s.

Lemma 3.9

Let E be a set of equations on $\mathrm{Term}(X)$, and let $\equiv_E$ be a relation on $\mathrm{Term}(X)$ defined by $s \equiv_E t \Leftrightarrow E \vdash s = t$. Then, the following hold:
(1) $\equiv_E$ is a congruence relation.
(2) For any homom. $\phi : \mathcal{T}(X) \to \mathcal{T}(X)$, $\quad s \equiv_E t \Rightarrow \phi(s) \equiv_E \phi(t)$.
(3) For any homom. $\phi : \mathcal{T}(X) \to \mathcal{T}(X)/{\equiv_E}$, there exists a hom. $\psi : \mathcal{T}(X) \to \mathcal{T}(X)$ s.t.

$$\phi = \pi_{\equiv_E} \circ \psi.$$

Lemma 3.10

$\mathcal{T}(X)/{\equiv_E}$ is the free $\mathrm{Mod}(E)$-algebra generated by $\pi_{\equiv_E}(X)$.

Note. This lemma also holds for any invariant congruence $\equiv$.

Proof.

Claim 1. $\mathcal{T}(X)/{\equiv_E} \in \mathrm{Mod}(E)$

- It suffices to show that for any equation $s = t$ in E and any homomorphism $\phi : \mathcal{T}(X) \to \mathcal{T}(X)/{\equiv_E}$, we have $\phi(s) = \phi(t)$.

Claim 2. $\mathcal{T}(X)/{\equiv_E}$ is a free algebra.

- For any $\mathfrak{A} \models E$ and $\phi : X/{\equiv_E} \to |\mathfrak{A}|$, by the corollary to the homomorphism theorem, there exists $\hat{\phi} : \mathcal{T}(X)/{\equiv_E} \to \mathfrak{A}$ s.t. $\hat{\psi} = \hat{\phi} \circ \pi_{\equiv_E}$, which is a unique homomorphism extending ϕ $\square$

Proof of the completeness theorem: Let $E \models s = t$. Since $\mathcal{T}(X)/{\equiv_E} \in \mathrm{Mod}(E)$, we have $\mathcal{T}(X)/{\equiv_E} \models s = t$. Then for any homomorphism $\phi : \mathcal{T}(X) \to \mathcal{T}(X)/{\equiv_E}$, we have $\phi(s) = \phi(t)$. In particular, letting $\phi = \pi_{\equiv_E}$, we have $s \equiv_E t$. Hence, $E \vdash s = t$.

Birkhoff's variety theorem

Definition 3.12

If a set $\mathcal{K}$ of $\mathcal{L}$-algebras is said to be an **equational class** (or **variety**) if it is characterized by a set E of equations, that is

$$\mathcal{K} = \mathrm{Mod}(E).$$

Theorem 3.13 (Birkhoff's variety theorem)

$\mathcal{K}$ is an equational class $\Leftrightarrow$ $\mathcal{K}$ is closed under subalgebras, homomorphisms, and Cartesian products.

Proof.

To show $\Rightarrow$

- It is clear since an equation that holds in some algebraic structure also holds in its subalgebras and homomorphic images.
- The equality that holds for each $\mathfrak{A}_i$ also holds for the Cartesian product $\prod \mathfrak{A}_i$.

To show $\Leftarrow$

- Let $\mathcal{K}$ be closed under subalgebras, homomorphisms, and Cartesian products.
- Let X be an infinite set of variables. We define the following set of equations in $\mathrm{Term}(X)$ as follows:

$$E = \{s = t : \text{for any } \mathfrak{A} \in \mathcal{K}, \mathfrak{A} \models s = t\}.$$

- Our aim is to show $\mathrm{Mod}(E) = \mathcal{K}$.
- $\mathrm{Mod}(E) \supseteq \mathcal{K}$ is obvious. Hence, we will prove the following by two steps.

Claim. $\mathrm{Mod}(E) \subseteq \mathcal{K}$.

The outline of the poof: Suppose $\mathfrak{A} \in \mathrm{Mod}(E)$. It suffices to construct a homomorphism from $\mathfrak{C} \in \mathcal{K}$ onto $\mathfrak{A}$.
Take a set Y of variables such that there exists a surjection $\chi : Y \to |\mathfrak{A}|$. Thus, there also exists an epimorphism (surjective homom.) $\hat{\chi} : \mathcal{T}(Y) \to \mathfrak{A}$.
Now, we are going to construct $\mathfrak{C} \in \mathcal{K}$ which intends to represent $\mathcal{T}(Y)/E$ by suitable replacement of variables.

For any $\mathfrak{B} \in \mathcal{K}$ and any homomorphism $\phi : \mathcal{T}(Y) \to \mathfrak{B}$, we define a congruence relation $\approx_\phi$ on $\mathcal{T}(Y)$ such that $s \approx_\phi t \Leftrightarrow \phi(s) = \phi(t)$.
By the homomorphism theorem, we have $\phi(\mathcal{T}(Y)) \simeq \mathcal{T}(Y)/\approx_\phi$. Since the left-hand side is a subalgebra of $\mathfrak{B} \in \mathcal{K}$, by assumption we have $\mathcal{T}(Y)/\approx_\phi \in \mathcal{K}$.

Let $\mathcal{D}$ be the set of congruence relations on $\mathcal{T}(Y)$ expressed as $\approx_\phi$ for some homomorphism ϕ. Since $\mathcal{K}$ is closed under Cartesian products, we have

$$\prod_{\approx \in \mathcal{D}} (\mathcal{T}(Y)/\approx) \in \mathcal{K}.$$

With a homom. $\pi_\approx : \mathcal{T}(Y) \to \mathcal{T}(Y)/\approx$ for each $\approx \in \mathcal{D}$, we can naturally define a homom.

$$\psi : \mathcal{T}(Y) \to \prod_{\approx \in \mathcal{D}} (\mathcal{T}(Y)/\approx).$$

Since $\mathcal{T}(Y)/\approx_\psi$ is isomorphic to a subalgebra of $\prod_{\approx \in \mathcal{D}}(\mathcal{T}(Y)/\approx)$, it also belongs to $\mathcal{K}$.
Here, we have: $s \approx_\psi t \Leftrightarrow \psi(s) = \psi(t) \Leftrightarrow$ for each $\approx \in \mathcal{D}$ $s \approx t \Leftrightarrow$ for all ϕ $\phi(s) = \phi(t) \Leftrightarrow$ for all $\mathfrak{B} \in \mathcal{K}$, $\mathfrak{B} \models s = t \Leftrightarrow s = t \in E$ (with suitable replacement of variables). Thus, $\mathcal{T}(Y)/\approx_\psi$ is a desired algebra $\mathfrak{C}$. □

Non-equational class

- Since the axiom system of the group given in last lecture consists only of equations, the class of all groups is considered to form an equality class.
- However, groups can also be characterized in other ways. For example, we may define a group $(G, \bullet)$ as a pair of set and a binary operation on it, satisfying

$$(x \bullet y) \bullet z = x \bullet (y \bullet z) \text{ and there exists } w \text{ such that}$$
$$\text{for any } x \ (w \bullet x = x \text{ and } \exists y (y \bullet x = w))$$

- The group according to this definition is not an equational class. In fact, $(\mathbb{N}, +)$ is a subalgebra of the group $(\mathbb{Z}, +)$, but it is not a group.
- Abelian groups, rings, R-modules, lattices, Boolean algebras can be treated as equational classes with sufficient operators.
- But integral domains (commutative rings with no zero factors other than 0) and fields cannot be axiomatized only with equations, no matter how many operators are added. For example, the Cartesian product $\mathfrak{Z} \times \mathfrak{Z}$ of the integral domain $\mathfrak{Z} = (\mathbb{Z}, +, \bullet \cdot, 0, 1)$ is not an integral domain (note: $(0,1) \bullet (1,0) = (0,0)$).

A simple extension of equational theory

- Consider a language including relational symbols $\mathtt{R}_1, \mathtt{R}_2, \ldots$ besides function symbols.
- An expression of the form $s = t$ or $\mathtt{R}(t_1, \ldots, t_n)$ is called an **atomic formula**. We will give a deduction system that handles atomic formulas.

Definition 3.14

Let T be a set of atomic formulas. A deduction system for T consists of the following:
(1) Atomic formulas belonging to T and equations of the form $t = t$ are axioms.
(2) The following five diagrams are inference rules.

$$\frac{s = t}{t = s} \qquad \frac{s = t \quad t = u}{s = u}$$

$$\frac{s(x) = t(x)}{s(u) = t(u)} \qquad \frac{s_1 = t_1 \ \ldots \ s_n = t_n}{\mathtt{f}(s_1, \ldots, s_n) = \mathtt{f}(t_1, \ldots, t_n)}$$

$$\frac{s_1 = t_1 \ \ldots \ s_n = t_n \quad \mathtt{R}(s_1, \ldots, s_n)}{\mathtt{R}(t_1, \ldots, t_n)}$$

where, s, t, s_i, t_i, u are terms, x is a variable, $\mathtt{f}$ is a function symbol, and $\mathtt{R}$ is a relation symbol.

- A proof tree for this system is defined in the same way as for an equational theory, and if an atomic formula σ has a proof tree, we write $T \vdash \sigma$.
- The concept of models of T and $T \models \sigma$ can be naturally defined by extending structures to involve interpretations of relational symbols.
- For this extended theory, the same assertions as Birkhoff's completeness theorem and variety theorem hold.

§4. Boolean Algebras

Boolean algebra is defined as an equational theory as follows.

Definition 4.3

The theory of **Boolean algebra** (BA) is defined in language $\mathcal{L}_{\mathrm{B}} = \{\vee, \wedge, \neg, 0, 1\}$ as

❶ the axioms of lattices:

$$\begin{array}{lll} \text{L1}: & x \vee x = x,\ x \wedge x = x & \text{[Idempotence]} \\ \text{L2}: & x \vee y = y \vee x,\ x \wedge y = y \wedge x & \text{[Commutativity]} \\ \text{L3}: & x \vee (y \vee z) = (x \vee y) \vee z,\ x \wedge (y \wedge z) = (x \wedge y) \wedge z & \text{[Associativity]} \\ \text{L4}: & (x \vee y) \wedge x = x,\ (x \wedge y) \vee x = x & \text{[Absorption]} \end{array}$$

❷ the distributive law:

$$(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z), \quad (x \wedge y) \vee z = (x \vee z) \wedge (y \vee z).$$

❸ $x \vee 0 = x, \quad x \vee (\neg x) = 1, \quad x \wedge 1 = x, \quad x \wedge (\neg x) = 0.$

A model of theory BA is called a **Boolean algebra**.

In the definition of Boolean algebra, (1) can be reduced to only L2.

Lemma 4.4 (Uniqueness of complement)

If $x \vee y = 1$ and $x \wedge y = 0$, then $y = \neg x$.

Lemma 4.5 (Elimination of double negation)

$\neg\neg x = x$.

Theorem 4.6 (Duality theorem)

For an equation φ in $\mathcal{L}_{\mathrm{B}} = \{\vee, \wedge, \neg, 0, 1\}$, let $\tilde{\varphi}$ denote the equation (dual equation) obtained from φ by interchanging $\vee$ with $\wedge$ and 0 with 1. Then $\mathrm{BA} \vdash \varphi \Leftrightarrow \mathrm{BA} \vdash \tilde{\varphi}$.

Theorem 4.7 (De Morgan's laws)

In BA, $\neg(x \vee y) = \neg x \wedge \neg y$, $\neg(x \wedge y) = \neg x \vee \neg y$ holds.

Theorem 4.8 (Stone's representation theorem)

For any Boolean algebra $\mathfrak{B}$, there exists a set X, $\mathfrak{B}$ can be embedded into the power set algebra $\mathfrak{P}(X)$. Especially, if $\mathfrak{B}$ is finite, it is isomorphic to $\mathfrak{P}(X)$.

Shannon's expansion (decomposition) theorem

- A Boolean expression $\varphi(x_1, x_2, \ldots, x_n)$, i.e., a term in $\mathcal{L}_{\mathrm{B}}$ with only variables $\{x_1, x_2, \ldots, x_n\}$, defines a function $f_\varphi : \{0,1\}^n \to \{0,1\}$. Such functions are called **Boolean functions**.

- We want to show that any function $f : \{0,1\}^n \to \{0,1\}$ can be expressed as f_φ with some Boolean expression φ. Moreover, if two Boolean expressions φ and ψ define the same function $f_\varphi = f_\psi$, then $\varphi = \psi$ is a theorem of BA.

Lemma 4.9 (Shannon's theorem)

$\mathrm{BA} \vdash \varphi(x_1, x_2, \ldots, x_n) \leftrightarrow (\varphi(0, x_2, \ldots, x_n) \land \neg x_1) \lor (\varphi(1, x_2, \ldots, x_n) \land x_1)$ [1].

Proof.

- Given a Boolean expression, we use de Morgan's laws and double negation elimination to push the negation symbols innermost so that each negation appears just before an variable. A Boolean expression in such a form is called a **negation normal form**.

- Transform a Boolean expression φ into the negation normal form, i.e., each negation appears just before an variable.

- Then, we can prove the lemma by induction on the number of operators $\lor$ and

Notation. $\varphi_1 \vee \varphi_2 \vee \cdots \vee \varphi_n$ is also written as $\bigvee_{i=1,\ldots,n} \varphi_i$.
Furthermore, we set $x^b = x$ if $b = 1$ and $x^b = \neg x$ if $b = 0$.

Theorem 4.10 (Disjunctive normal form)

For a Boolean expression $\varphi(x_1, x_2, \ldots, x_n)$,

$$\begin{aligned}
\mathrm{BA} \vdash \varphi(x_1, x_2, \ldots, x_n) &= \bigvee_{b_1,\ldots,b_n=0,1} \varphi(b_1, b_2, \ldots, b_n) \wedge x_1^{b_1} \wedge x_2^{b_2} \wedge \cdots \wedge x_n^{b_n} \\
&= \bigvee_{f_\varphi(b_1,\ldots,b_n)=1} x_1^{b_1} \wedge x_2^{b_2} \wedge \cdots \wedge x_n^{b_n}.
\end{aligned}$$

If there is no $b_1, \ldots, b_n$ such that $f_\varphi(b_1, \ldots, b_n) = 1$, then we set the right-hand side $= 0$.

Proof By Shannon's theorem, we can prove this by induction on the number of variables. □

In addition, if we rewrite $\neg\sigma$ into the disjunctive normal form, then we can easily obtain a conjunctive normal form of σ by de Morgan's laws and double negation elimination.

Corollary 4.11

For any function $f : \{0,1\}^n \to \{0,1\}$, there exists a Boolean expression φ such that $f = f_\varphi$.

Proof. Obvious from the theorem ☐

Corollary 4.12

If two Boolean expressions φ and ψ define the same function $f_\varphi = f_\psi$, then $\mathrm{BA} \vdash \varphi = \psi$.

Proof. In the theorem, both disjunctive normal forms are the same. ☐

Corollary 4.13

The number of equivalence classes of Boolean expressions of n variables is 2^{2^n}.

Proof. The number of equivalence classes of a Boolean expression with n variable is equal to the number of the function $f : \{0,1\}^n \to \{0,1\}$, that is, 2^{2^n}. ☐

Finally, we introduce Boolean rings, which are essentially equivalent to Boolean algebras.

Definition 4.14

The theory CR of **commutative ring** consists of the following axioms, in the language $\mathcal{L}_{\mathrm{R}} = \{+, \bullet, -, 0, 1\}$.

$$x + 0 = x, \quad x + y = y + x, \quad x + (y + z) = (x + y) + z, \quad x + (-x) = 0,$$
$$x \bullet 1 = x, \quad x \bullet y = y \bullet x, \quad x \bullet (y \bullet z) = (x \bullet y) \bullet z, \quad x \bullet (y + z) = (x \bullet y) + (x \bullet z).$$

A model of the theory CR is called a **commutative ring**.

In BA and CR, we usually assume $0 \neq 1$ as an axiom. But since we want to treat them as an equational theory, we treat a structure where $0 = 1$ as a special case.

Example 13

The structure of integers $\mathfrak{Z} = (\mathbb{Z}, +, \bullet, -, 0, 1)$ is a commutative ring.

Example 14

For a commutative ring $\mathfrak{A}$, the set of polynomials with variables $X_1, X_2, \ldots, X_n$ and coefficients in A also becomes a commutative ring, denote $\mathfrak{A}[X_1, X_2, \ldots, X_n]$.

Definition 4.15

The theory BR of **Boolean rings** is the theory CR plus the following axiom.

$$x^2 = x.$$

A model of the theory BR is called a **Boolean ring**.

We first show that $x + x = 0$ holds in BR.

$$x + x = (x + x)^2 = x^2 + x^2 + x^2 + x^2 = x + x + x + x.$$

By subtracting $x + x$ from both sides, we get $x + x = 0$. So, $+$ in a Boolean ring has a different property from $+$ in a Boolean algebra. However, both are mutually translatable as shown in the next theorem.

Theorem 4.16 (Stone theorem)

(1) For any Boolean algebra $\mathfrak{B} = (B, \vee, \wedge, \neg, 0, 1)$, we set

$$x + y = (x \wedge (\neg y)) \vee ((\neg x) \wedge y), \quad x \bullet y = x \wedge y, \quad -x = x.$$

Then, $\mathfrak{B}^\circ = (B, +, \bullet, -, 0, 1)$ is a Boolean ring.

(2) For any Boolean ring $\mathfrak{R} = (R, +, \bullet, -, 0, 1)$, we set

$$x \vee y = x + y + x \bullet y, \quad x \wedge y = x \bullet y, \quad \neg x = 1 + x$$

and then $\mathfrak{R}^\circ = (R, \vee, \wedge, \neg, 0, 1)$ is a Boolean algebra.

(3) By (1) and (2), for a Boolean algebra $\mathfrak{B}$ and a Boolean ring $\mathfrak{R}$,

$$\mathfrak{B}^{\circ\circ} = \mathfrak{B},$$

$$\mathfrak{R}^{\circ\circ} = \mathfrak{R}.$$

Propositional logic

- Comparing with Boolean algebras, we will outline **propositional logic** which studies the logical relationships between propositions in terms of **propositional connectives**:
  $\neg$ (not $\cdots$), $\wedge$ (and), $\vee$ (or), $\to$ (implies).

- Propositions are constructed from atomic propositions $p, q, \ldots$ by way of propositional connectives. Atomic propositions are simply symbols that can take value either T (true) or F (false).

- Let v be a function that assigns truth values T or F to atomic propositions. Then, a **truth value assignment** V for all propositions are uniquely defined as follows.
  (1) for an atomic proposition φ, $V(\varphi) = v(\varphi)$.
  (2a) $V(\neg\varphi) = \mathrm{T} \overset{\text{def}}{\Longleftrightarrow} V(\varphi) = \mathrm{F}$,
  (2b) $V(\varphi \wedge \psi) = \mathrm{T} \overset{\text{def}}{\Longleftrightarrow} V(\varphi) = \mathrm{T}$ and $V(\psi) = \mathrm{T}$,
  (2c) $V(\varphi \vee \psi) = \mathrm{T} \overset{\text{def}}{\Longleftrightarrow} V(\varphi) = \mathrm{T}$ or $V(\psi) = \mathrm{T}$,
  (2d) $V(\varphi \to \psi) = \mathrm{T} \overset{\text{def}}{\Longleftrightarrow} V(\varphi) = \mathrm{F}$ or $V(\psi) = \mathrm{T}$.

Definition 4.17

If a proposition φ is always true, i.e., $V(\varphi) = \mathrm{T}$ for any truth-value function V, then φ is said to be **valid** or a **tautology**, written as $\models \varphi$.

- We consider the structure of the tautologies.
- To this end, it is not necessary to deal with all four propositional symbols at once. By setting $\varphi \to \psi := \neg\varphi \vee \psi$, we omit $\to$.

Definition 4.18

- A finite sequence of propositions $\varphi_1, \ldots, \varphi_n$ is called a **sequent**.
- All sequents formed by rearranging the elements are regarded as the same.
- A sequent is therefore a multiset rather than a sequence.
- For two sequents Γ $(= \varphi_1, \ldots, \varphi_n)$ and Δ $(= \psi_1, \ldots, \psi_m)$, Γ, Δ denotes the sequent $\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_m$.

The sequent $\varphi_1, \ldots, \varphi_n$ intuitively means $\varphi_1 \vee \cdots \vee \varphi_n$.

Formal deductive system

Definition 4.19 (Formal deductive system)

The **Gentzen-style system** of propositional logic has the following axioms and rules.

Axioms: $\neg\psi, \psi$ (Law of excluded middle)

Inference rules:

$$\frac{\Gamma, \varphi, \psi}{\Gamma, \varphi \vee \psi}\ (\vee), \quad \frac{\Gamma, \varphi \quad \Gamma, \psi}{\Gamma, \varphi \wedge \psi}\ (\wedge)$$

$$\frac{\Gamma, \neg\varphi, \neg\psi}{\Gamma, \neg(\varphi \wedge \psi)}\ (\neg\wedge), \quad \frac{\Gamma, \neg\varphi \quad \Gamma, \neg\psi}{\Gamma, \neg(\varphi \vee \psi)}\ (\neg\vee)$$

$$\frac{\Gamma}{\Delta}\ (\mathrm{weak}^2)\ (\text{if } \Gamma \subset \Delta), \quad \frac{\Gamma, \neg A \quad \Gamma, A}{\Gamma}\ (\mathrm{cut})$$

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[2] weakening rule

Definition 4.20

- A **proof tree** is a finite tree in which each vertex is labelled with a sequent so that a sequent at each top vertex (leaf) is an axiom, and the sequents of adjacent nodes express an inference rule. See the example below.
- If there is a proof tree rooted at the sequent Γ, we write it as $\vdash \Gamma$. Such a tree is called a **proof** of $\vdash \Gamma$ (equivalently, a **proof** of the disjunction of all propositions in Γ).

Example

A proof of $\vdash \neg\varphi \vee (\neg\psi \vee (\varphi \wedge \psi))$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg\varphi,\varphi}{\neg\varphi,\varphi,\neg\psi}\,(\text{weak}) \quad \dfrac{\neg\psi,\psi}{\neg\psi,\psi,\neg\varphi}\,(\text{weak})}{\neg\varphi,\neg\psi,\varphi\wedge\psi}\,(\wedge)}{\neg\varphi,\neg\psi\vee(\varphi\wedge\psi)}\,(\vee)}{\neg\varphi\vee(\neg\psi\vee(\varphi\wedge\psi))}\,(\vee)$$

Induced rules and completeness theorems

We next show that the reversal of rule $(\wedge)$ can be used as a new rule ($\wedge$-elim).

$$\dfrac{\dfrac{\varphi\wedge\psi}{\varphi,\varphi\wedge\psi}\,(\text{weak}) \qquad \dfrac{\dfrac{\varphi,\neg\varphi}{\varphi,\neg\varphi,\neg\psi}\,(\text{weak})}{\varphi,\neg(\varphi\wedge\psi)}\,(\neg\wedge)}{\varphi}\,(\text{cut}) \qquad \Rightarrow \qquad \dfrac{\Gamma,\varphi\wedge\psi}{\Gamma,\varphi}\,(\wedge\text{-elim})$$

Similarly, we can provide a new rule ($\vee$-elim).

The completeness theorem for propositional logic

$\vdash\varphi \;\Leftrightarrow\; \models\varphi$.

Homework Problem # 4

Consider the relation between the completeness theorem for propositional logic and that for Boolean algebra. Hint: (prop. logic) $\vdash\varphi \;\Leftrightarrow\; \mathrm{BA}\vdash\varphi=1$.

Thank you for your attention!