Logic and Computation I

Part 3. First order logic and decision problems

Kazuyuki Tanaka

BIMSA

November 19, 2024

北京雁栖湖应用数学研究院
Beijing Institute of Mathematical Sciences and Applications

Logic and Computation I

- **Part 1. Introduction to Theory of Computation**
- **Part 2. Propositional Logic and Computational Complexity**
- **Part 3. First Order Logic and Decision Problems**
- **Part 4. Modal logic** (shifted to next semester)

Part 3. A new schedule (subject to change)

- Nov.14, (4) Ehrenfeucht-Fraïssé's theorem
- Nov.19, (5) Ehrenfeucht-Fraïssé's theorem II
- Nov.21, (6) Presburger arithmetic
- Nov.26, (7) Peano arithmetic
- Nov.28, (8) Gödel's first incompleteness theorem
- Dec. 3, (9) Gödel's second incompleteness theorem
- Dec. 5, (10) Second order logic
- Dec.10, (11) Second order arithmetic

Recap

- We will consider a language of finitely many relation symbols and constants. Let $\mathcal{L}$ be $\{\mathrm{R}_0, \dots, \mathrm{R}_{n-1}\}$, and also consider its expansion by adding constants.
- Let $\mathcal{A}$ be a structure in a language $\mathcal{L} = \mathcal{L}' \cup \{\vec{c}\}$. Then, $\mathcal{A}$ can be written as $(\mathcal{A}', \vec{a})$ or $\mathcal{A}'_{\vec{a}}$, where $\mathcal{A}'$ is a reduct of $\mathcal{A}$ to the language $\mathcal{L}'$. Also, $\mathcal{A}$ can be expanded to $(\mathcal{A}, \vec{b}) = \mathcal{A}'_{\vec{a},\vec{b}}$ in a language $\mathcal{L}' \cup \{\vec{c}, \vec{d}\}$.
- The **(quantifier) rank** of a formula φ expresses the maximum number of nesting quantifiers, e.g., the rank of $\forall y(\forall x \exists y(x = y) \wedge \forall z(z > 0))$ is 3.
- **Lemma 3.18** For an n, k, there are essentially finitely many formulas with rank $\leq n$ in fixed free variables $x_1, ..., x_k$.

- The **theory** $\mathrm{Th}(\mathcal{A})$ of a structure $\mathcal{A}$ is the set of sentences true in $\mathcal{A}$.
- Two structures $\mathcal{A}, \mathcal{B}$ are said to be **elementarily equivalent**, denote $\mathcal{A} \equiv \mathcal{B}$, iff $\mathrm{Th}(\mathcal{A}) = \mathrm{Th}(\mathcal{B})$, equivalently $\mathcal{B} \models \mathrm{Th}(\mathcal{A})$.
- $\mathcal{A}$ is an **elementary substructure** of $\mathcal{B}$, $\mathcal{A} \prec \mathcal{B}$, iff $\mathrm{Th}(\mathcal{A}_A) = \mathrm{Th}(\mathcal{B}_A)$.
- Let $\mathrm{Th}_n(\mathcal{A})$ be the subset of $\mathrm{Th}(\mathcal{A})$ consisting of sentences with rank $\leq n$. A relation $\equiv_n$ is defined as follows.

$$\mathcal{A} \equiv_n \mathcal{B} \Leftrightarrow \mathrm{Th}_n(\mathcal{A}) = \mathrm{Th}_n(\mathcal{B}).$$

- Let $\mathcal{A}, \mathcal{B}$ be $\mathcal{L}$-structures with/without constants. Then, a function $f : \vec{a}(\subset A) \to \vec{b}(\subset B)$ is a **partial isomorphism** such that

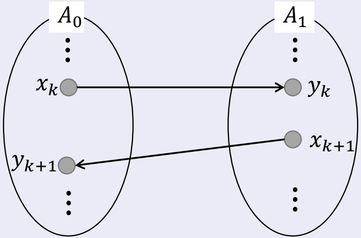
$$(\mathcal{A}, \vec{a}) \equiv_0 (\mathcal{B}, \vec{b}).$$

Note that if $\mathcal{A} \equiv_0 \mathcal{B}$, the empty function is a partial isomorphism. Moreover, notice that $\mathrm{Th}_0(\mathcal{A})$ and $\mathrm{Th}_0(\mathcal{B})$ may be empty without constants.

Definition 3.22

Let $\mathcal{A}_0$, $\mathcal{A}_1$ be $\mathcal{L}$-structures with/without constants and n be a natural number. In an n-round **EF game** $\mathrm{EF}_n(\mathcal{A}_0, \mathcal{A}_1)$, player I (Spoiler) and player II (Duplicator) alternately choose a number from A_0 or A_1 obeying the rules described below, and the winner is determined according to the winning condition.

- **Rules**: At each round $k \leq n$, if I chooses an element, say x_k, from A_i $(i = 0, 1)$, then II chooses an element, say y_k, from A_{1-i}.
- **Winning conditions**: If the correspondence $x_i \leftrightarrow y_i$ chosen by the players up to n rounds determines a partial isomorphism between $\mathcal{A}_0$ and $\mathcal{A}_1$, then II wins.



Definition 3.23

(With/without constants) $\mathcal{A} \simeq^n \mathcal{B}$ iff player II has a winning strategy in $\mathrm{EF}_n(\mathcal{A}, \mathcal{B})$.

Lemma 3.24

Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{L}$-structures. For each $n \geq 0$, we define a relation $\simeq^n$ as follows:

$$(\mathcal{A}, \vec{a}) \simeq^0 (\mathcal{B}, \vec{b}) \Leftrightarrow (\mathcal{A}, \vec{a}) \equiv_0 (\mathcal{B}, \vec{b}) \Leftrightarrow \vec{a} \mapsto \vec{b} \text{ is partial isomorphism.}$$

$$\begin{aligned}(\mathcal{A}, \vec{a}) \simeq^{n+1} (\mathcal{B}, \vec{b}) \Leftrightarrow\ & \forall a \in A\ \exists b \in B\ (\mathcal{A}, \vec{a}a) \simeq^n (\mathcal{B}, \vec{b}b) \text{ and} \\ & \forall b \in B\ \exists a \in A\ (\mathcal{A}, \vec{a}a) \simeq^n (\mathcal{B}, \vec{b}b)\end{aligned}$$

For $n = 0$, $(\mathcal{A}, \vec{a}) \simeq^0 (\mathcal{B}, \vec{b})$ iff II wins $\mathrm{EF}_0((\mathcal{A}, \vec{a}), (\mathcal{B}, \vec{b}))$ iff $(\mathcal{A}, \vec{a}) \equiv_0 (\mathcal{B}, \vec{b})$.

For the induction step, note that II wins $\mathrm{EF}_{n+1}(\mathcal{A}, \mathcal{B})$ iff

$\forall a \in A\ \exists b \in B$ II wins $\mathrm{EF}_n((\mathcal{A}, a), (\mathcal{B}, b))$ and

$\forall b \in B\ \exists a \in A$ II wins $\mathrm{EF}_n((\mathcal{A}, a), (\mathcal{B}, b))$.

Here, “II wins G” means “II has a winning strategy in G”.

Our goal is to show that $\mathcal{A} \simeq^n \mathcal{B}$ and $\mathcal{A} \equiv_n \mathcal{B}$ are equivalent. To this end, we introduce the Scott-Hintikka formulas.

Definition 3.25 (Scott-Hintikka Formula)

For a structure $\mathcal{A}$ and a sequence of elements $\vec{a}$, the **Scott-Hintikka formula** with rank n, $\varphi^n_{\mathcal{A},\vec{a}}(\vec{x})$, is defined inductively as follows.

$$\varphi^0_{\mathcal{A},\vec{a}}(\vec{x}) := \bigwedge \{\theta(\vec{x}) : (\mathcal{A},\vec{a}) \models \theta(\vec{c}),\ \mathrm{qr}(\theta(\vec{x})) = 0\}\,.$$

$$\varphi^{n+1}_{\mathcal{A},\vec{a}}(\vec{x}) := \bigwedge_{a\in A} \exists x\ \varphi^n_{\mathcal{A},\vec{a}a}(\vec{x},x) \wedge \forall x \bigvee_{a\in A} \varphi^n_{\mathcal{A},\vec{a}a}(\vec{x},x).$$

- When we write $(\mathcal{A},\vec{a}) \models \theta(\vec{c})$, $\vec{c}$ are new constants interpreted as $\vec{a}$.
- In the above definition, even if A is infinite, by Lemma 3.18, there are finitely many formulas in the scopes of $\bigwedge, \bigvee$. So, the Scott-Hintikka formula can be defined as a first-order formula.

Lemma 3.26

$(\mathcal{A}, \vec{a}) \models \varphi^n_{\mathcal{A},\vec{a}}(\vec{c})$.

Proof

- When $n = 0$, it is clear from the definition.
- Then, we want to show $(\mathcal{A}, \vec{a}) \models \varphi^{n+1}_{\mathcal{A},\vec{a}}(\vec{c})$ by the induction hypothesis.
- We first consider $\bigwedge_{a \in A} \exists x \varphi^n_{\mathcal{A},\vec{a}a}(\vec{c}, x)$, which is the left component of the definition formula of $\varphi^{n+1}_{\mathcal{A},\vec{a}}(\vec{c})$. For every $a \in A$, $\varphi^n_{\mathcal{A},\vec{a}a}(\vec{c}, c)$ holds in $(\mathcal{A}, \vec{a}a)$ by the induction hypothesis. So, $\exists x \varphi^n_{\mathcal{A},\vec{a}a}(\vec{c}, x)$ also holds in $(\mathcal{A}, \vec{a}a)$, hence also in $(\mathcal{A}, \vec{a})$. Finally, the left formula holds for $(\mathcal{A}, \vec{a})$.
- To show the right formula $\forall x \bigvee_{a \in A} \varphi^n_{\mathcal{A},\vec{a}a}(\vec{c}, x)$ holds in $(\mathcal{A}, \vec{a})$, take any $x = b \in A$. Then, letting $a = b$, we may show $\varphi^n_{\mathcal{A},\vec{a}b}(\vec{c}, c)$ holds in $(\mathcal{A}, \vec{a}b)$ (where $c = b$), which holds by the induction hypothesis. So, the right formula also holds for $(\mathcal{A}, \vec{a})$.
- Therefore, the conjunction of both formulas holds in $(\mathcal{A}, \vec{a})$. □

Theorem 3.27 (Ehrenfeucht-Fraïss theorem, EF theorem)

For all $n \geq 0$, the following are equivalent.
(1) $(\mathcal{A},\vec{a}) \equiv_n (\mathcal{B},\vec{b})$, (2) $(\mathcal{B},\vec{b}) \models \varphi^n_{\mathcal{A},\vec{a}}(\vec{c})$, (3) $(\mathcal{A},\vec{a}) \simeq^n (\mathcal{B},\vec{b})$.

Proof. (1) $\Rightarrow$(2). It is obvious from Lemma 3.26, since $\mathrm{qr}(\varphi^n_{\mathcal{A},\vec{a}}(\vec{x})) = n$.
We show (2) $\Rightarrow$(3) by induction on n. For $n = 0$, (2)$\Rightarrow (\mathcal{A},\vec{a}) \equiv_0 (\mathcal{B},\vec{b}) \Rightarrow (3)$.
For induction step, assume (2) $\Rightarrow$(3) for n as well as $(\mathcal{B},\vec{b}) \models \varphi^{n+1}_{\mathcal{A},\vec{a}}(\vec{c})$.
From the definition of the Scott-Hintikka formula,

$$\forall a \in A\ \exists b \in B\ (\mathcal{B},\vec{b}b) \models \varphi^n_{\mathcal{A},\vec{a}a}(\vec{c},c)\ \wedge\ \forall b \in B\ \exists a \in A\ (\mathcal{B},\vec{b}b) \models \varphi^n_{\mathcal{A},\vec{a}a}(\vec{c},c)$$

By the induction hypothesis, we have

$$\forall a \in A\ \exists b \in B\ (\mathcal{A},\vec{a}a) \simeq^n (\mathcal{B},\vec{b}b)\ \wedge\ \forall b \in B\ \exists a \in A\ (\mathcal{A},\vec{a}a) \simeq^n (\mathcal{B},\vec{b}b).$$

By Lemma 3.24, we obtain

$$(\mathcal{A},\vec{a}) \simeq^{n+1} (\mathcal{B},\vec{b}).$$

Thus, (3) also holds for $n+1$.

We finally show (3) $\Rightarrow$(1) by induction on n.
Case $n = 0$ follows from Lemma 3.24.
For induction step, assume (3) $\Rightarrow$ (1) for n as well as $(\mathcal{A}, \vec{a}) \simeq^{n+1} (\mathcal{B}, \vec{b})$.

- To show $(\mathcal{A}, \vec{a}) \equiv_{n+1} (\mathcal{B}, \vec{b})$, the essential case to check is a formula $\varphi(\vec{x}) = \exists x \psi(\vec{x}, x)$ with $\mathrm{qr}(\psi(\vec{x}, x)) = n$.
- Suppose $(\mathcal{A}, \vec{a}) \models \varphi(\vec{c})$. Then, there exists $a \in A$ such that $(\mathcal{A}, \vec{a}a) \models \psi(\vec{c}, c)$.
- Since $(\mathcal{A}, \vec{a}) \simeq^{n+1} (\mathcal{B}, \vec{b})$, by Lemma 3.24, there exists a $b \in B$ such that $(\mathcal{A}, \vec{a}a) \simeq^{n} (\mathcal{B}, \vec{b}b)$, and so $(\mathcal{B}, \vec{b}b) \models \psi(\vec{c}, c)$. Thus $(\mathcal{B}, \vec{b}) \models \varphi(\vec{c})$.
- This proves $\mathrm{Th}_{n+1}(\mathcal{A}, \vec{a}) \subset \mathrm{Th}_{n+1}(\mathcal{B}, \vec{b})$. Similarly, we have $\mathrm{Th}_{n+1}(\mathcal{A}, \vec{a}) \supset \mathrm{Th}_{n+1}(\mathcal{B}, \vec{b})$, and so (1) holds. □

Corollary 3.28

$\mathcal{A} \equiv \mathcal{B} \Leftrightarrow$ for any n, $\mathcal{A} \simeq^{n} \mathcal{B}$.

It is natural to extend the play of the EF game to infinity (ω-round). If player II has a winning strategy in such a game $\mathrm{EF}_\omega(\mathcal{A}, \mathcal{B})$, we write $\mathcal{A} \simeq^\omega \mathcal{B}$.

Corollary 3.29

Suppose $\mathcal{A}, \mathcal{B}$ are countable. Then, $\mathcal{A} \simeq^\omega \mathcal{B} \ \Leftrightarrow \ \mathcal{A} \simeq \mathcal{B}$.

Proof. $\Leftarrow$ is obvious because the isomorphism is a winning strategy for player II. $\Rightarrow$ is shown by the **back-and-forth argument**. Let $A = \{a_0, a_1, \dots\}$, $B = \{b_0, b_1, \dots\}$. Player II follows the winning strategy, and Player I alternately chooses the smallest element that have not been selected from A and B, thus a bijection between $\mathcal{A}$ and $\mathcal{B}$ is produced, which is a desired isomorphism. □

Corollary 3.30

For each n, there are finitely many equivalence classes of $\mathcal{L}$-structure by $\equiv_n$.

Proof By Lemma 3.18, there are essentially finitely many Scott-Hintikka sentences $\varphi^n_{\mathcal{A},\varnothing}$ with rank n. By the EF theorem, each $\equiv_n$ equivalence class is characterized by such a sentence, and so there are only a finite number of them. □

Corollary 3.31

Let K be a set of $\mathcal{L}$-structures. The following are equivalent.

(1) For any n, there exist $\mathcal{A} \in K$ and $\mathcal{B} \notin K$ such that $\mathcal{A} \equiv_n \mathcal{B}$.

(2) K is not an elementary class (K cannot be defined by a first-order formula).

Proof.

- (1) $\Rightarrow$(2). By way of contradiction, assume K is defined by a first-order sentence φ. Let n be the rank of φ. If $\mathcal{A} \in K$ and $\mathcal{B} \notin K$ then $\mathcal{A} \not\equiv_n \mathcal{B}$.
- (2) $\Rightarrow$(1). By way of contradiction, assume that for some n, if $\mathcal{A} \equiv_n \mathcal{B}$ then $\mathcal{A} \in K \Leftrightarrow \mathcal{B} \in K$. Since there is a first-order (Scott-Hintikka) sentence $\varphi^n_{\mathcal{A}}$ of rank n such that $\mathcal{A} \equiv_n \mathcal{C} \Leftrightarrow \mathcal{C} \models \varphi^n_{\mathcal{A}}$, K is defined by $\varphi^n_{\mathcal{A}}$. □

NOTE: Definition 3.32 and Theorem 3.33 are skipped and will be explained later.

Applications of EF games: DLO

- $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$ are models of DLO (dense linear order without end points).
- Let $\mathcal{A}, \mathcal{B}$ be two models of DLO. Player II has a winning strategy in $\mathrm{EF}_n(\mathcal{A}, \mathcal{B})$ for all n. Suppose a partial isomorphism between $a_1 < a_2 < \cdots < a_n$ in A and $b_1 < b_2 < \cdots < b_n$ in B are constructed by the players up to the round n. If Player I chooses x_{n+1} between $a_i < a_{i+1}$ (or $b_i < b_{i+1}$), then Player II can extend the partial isomorphism by choosing y_{n+1} between $b_i < b_{i+1}$ (or $a_i < a_{i+1}$).
- Then, for all $n \geq 0$, $\mathcal{A} \simeq^n \mathcal{B}$. By the EF theorem, for all n, $\mathcal{A} \equiv_n \mathcal{B}$, and hence $\mathcal{A} \equiv \mathcal{B}$. In particular, $(\mathbb{Q}, <) \equiv (\mathbb{R}, <)$.
- Then, DLO is a complete theory. Therefore, it is decidable.
  - If it is not complete, then there is a sentece σ which is neither provable nor disprovable.
  - Hence, both $\mathrm{DLO} \cup \{\neg\sigma\}$ and $\mathrm{DLO} \cup \{\sigma\}$ are consistent. So, each has its own model, but they are no longer elementary equivalent, which is a contradiction.

Theorem 3.34

DLO is a PSPACE-complete problem.

Proof. First, we show that DLO is PSPACE-hard, by reducing TQBF to DLO in polynomial time. It was shown in Part 2 of this course, TQBF (true quantified Boolean formula) is PSPACE-complete.

- Let A be a QBF and transform it to a PNF $Q_1 x_1 \ldots Q_n x_n B(x_1, \ldots, x_n)$, where $B(x_1, \ldots, x_n)$ is a Boolean formula.
- Then, define a DLO formula $A_<$ as follows.

$$Q_1 x_1 Q_1 y_1 \ldots Q_n x_n Q_n y_n B(x_1 < y_1, \ldots, x_n < y_n).$$

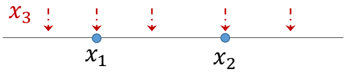
- For example, for a QBF $A \equiv \forall x_1 \exists x_2 \forall x_3 ((x_1 \wedge x_2) \vee \neg x_3)$, $A_<$ in DLO is

$$\forall x_1 \forall y_1 \exists x_2 \exists y_2 \forall x_3 \forall y_3 (((x_1 < y_1) \wedge (x_2 < y_2)) \vee \neg (x_3 < y_3)).$$

- An atomic formula $x_i < y_i$ in $A_<$ simply plays the role of variable x_i in A. So, A holds in the Boolean algebra $\{0, 1\}$ iff $A_<$ holds in any model of DLO.
- Since the lengths of A and $A_<$ differ only by constant multiples, TQBF is reduced to DLO in polynomial time.

Next, we show that DLO is PSPACE, following the proof that TQBF is PSPACE.

- First, assume a DLO formula is given in PNF $Q_1x_1 \ldots Q_nx_n\ C(x_1, \ldots, x_n)$ (with no quantifiers in $C(x_1, \ldots, x_n)$).
- To determine the truth value of $C(x_1, \ldots, x_n)$, only the relation $<$ among the elements are necessary. We first fix x_1 is arbitrarily. Next, the necessary information on x_2 is whether it is larger, smaller, or equal to x_1.
- If Q_2 is $\forall$ ($\exists$), all the three cases (one of the three cases) must hold. Without loss of generality, we may assume $x_1 < x_2$.
- Next, there are five cases for x_3 as illustrated by the red arrows:



  So, if Q_3 is $\forall$ ($\exists$), all the five cases (one of the five cases) should hold.
- In order to execute this computation, we need $\log((2n-1)!) = O(n \log n)$ space to keep records. Thus, it is DSPACE$(n \log n)$, hence also PSPACE. □

We next apply the EF theorem to the problem of length of finite linear orders.

Lemma 3.35 (Gurevich)

Fix any $m > 0$. If L_1, L_2 are two finite linear orders with length $\geq 2^m$, $L_1 \equiv_m L_2$.

Proof.

- By $[n] = (n, <)$, we denote a finite linear order on n, where n is identified with $\{0, 1, \ldots, n-1\}$.
- For each k, we define a threshold function $|x|_k$ by $|x|_k = |x|$ if $|x| < 2^k$; $|x|_k = \infty$, otherwise.
- Consider a partial isomorphism $\vec{a}(\subset [n]) \mapsto \vec{b}(\subset [n'])$ that satisfies the following conditions: if $\vec{a} = (a_1, a_2, \ldots, a_l)$ and $\vec{b} = (b_1, b_2, \ldots, b_l)$ are arranged in ascending order, and $a_0 = b_0 = 0$, $a_{l+1} = n$, $b_{l+1} = n'$, then
  for any $i \leq l$, $|a_{i+1} - a_i|_k = |b_{i+1} - b_i|_k$ holds.
  Then, let I_k be the set of such partial isomorphisms.
- By $\varnothing \in I_k$ we mean $|n|_k = |n'|_k$. Thus, if $n, n' \geq 2^m$, then $\varnothing \in I_m$

- Take any $\vec{a} \mapsto \vec{b} \in I_k$. We can show that for any $a \in n$, there exists a $b \in n'$ such that $\vec{a}a \mapsto \vec{b}b \in I_{k-1}$ holds. Here, $\vec{a}a$ and $\vec{b}b$ are rearranged in order.
- First consider the case $|a_{i+1} - a_i|_k = |b_{i+1} - b_i|_k < \infty$ and $a_{i+1} > a > a_i$. Then, $|a_{i+1} - a|_{k-1} < \infty$ or $|a - a_i|_{k-1} < \infty$ hold. For instance, if $|a - a_i|_{k-1} = d < \infty$, then $a = a_i + d$ and we may take $b = b_i + d$.
- Next consider the case $|a_{i+1} - a_i|_k = |b_{i+1} - b_i|_k = \infty$ and $a_{i+1} > a > a_i$. Then $|a_{i+1} - a|_{k-1} = \infty$ or $|a - a_i|_{k-1} = \infty$ holds. If one is $< \infty$, then b is determined in the same way as above. If both are ∞, b can be taken so that $|b_{i+1} - b|_{k-1} = \infty$ and $|b - b_i|_{k-1} = \infty$.
- Therefore, we have $I_0 \neq \varnothing$. More strictly, we obtain $[n] \simeq^m [n']$.
- Thus, by the EF theorem, for $n, n' \geq 2^m$, $[n] \equiv_m [n']$. □

Theorem 3.36

There is no first-order formula expressing the parity of length of a finite linear order.

Proof Assume we have such a formula φ. Let $\mathrm{qr}(\varphi) = m$. Then by the above lemma, linear orders longer than 2^m cannot be separated by φ, a contradiction. □

The connectivity of graphs cannot be defined by a first-order formula.

- We show this by reducing the parity problem of linear orders to it. We first make a special graph from a linear order.
- Given a linear order $<$, let $\mathrm{succ}(x,y) \equiv (x<y) \wedge \forall z(z \leq x \vee y \leq z)$ and $\mathrm{succ2}(x,y) \equiv \exists z(\mathrm{succ}(x,z) \wedge \mathrm{succ}(z,y))$.
  Also let $\mathrm{first}(x) \equiv \neg\exists y\ \mathrm{succ}(y,x)$ and $\mathrm{last}(x) \equiv \neg\exists y\ \mathrm{succ}(x,y)$
- Finally, we make a graph on $V = n$ by defining $\mathrm{edge}(x,y)$ as follows.
  $$\mathrm{edge}(x,y) \equiv \mathrm{succ2}(x,y) \vee$$
  $$((\exists z(\mathrm{succ}(x,z) \wedge \mathrm{last}(z)) \wedge \mathrm{first}(y))) \vee (\mathrm{last}(x) \wedge (\exists z(\mathrm{first}(z) \wedge \mathrm{succ}(z,y))))$$
  In this graph, every other points in a line are connected by an edge, and the first point is connected from the second last point, and also the second point is from the last point.
- If a linear order has even number of points, the graph becomes two cycles (disconnected), and if odd number, it results in a single cycle.
- In other words, if the connectivity of a graph can be defined, then the parity of the length of a linear order can be defined, a contradiction.

Homework 3.5.1

Given a finitely connected graph, the existence of an Eulerian cycle in it cannot be described in first-order logic.

- To expand the scope of application of the EF theorem, we would like to consider structures with functions.
- Rewriting functions as relations requires the use of extra quantifiers for function composition, and the need to use more complicated formulas for atomic formulas involving functions.
- However, there are no big problems when dealing with arbitrary ranks. For example, the following argument is possible for groups.
- $G_1 \equiv G_2 \Rightarrow G_1 \times H \equiv G_2 \times H$ for three groups G_1, G_2, H. For this proof, we observe that II's winning play $\vec{g_1} \leftrightarrow \vec{g_2}$ in $\mathrm{EF}_n(G_1, G_2)$ can be modified as II's winning play $(\vec{g_1}, \vec{h}) \leftrightarrow (\vec{g_2}, \vec{h})$ in $\mathrm{EF}_n(G_1 \times H, G_2 \times H)$.

Summary

- We consider a language of finitely many relation symbols and constants.
- By $\mathcal{A} \equiv_n \mathcal{B}$, we mean that $\mathcal{A}, \mathcal{B}$ satisfy the same formulas with rank $\leq n$.
- Let $\varphi^n_{\mathcal{A}}$ be the **Scott-Hintikka sentence** of rank n. Then $\mathcal{C} \equiv_n \mathcal{A} \Leftrightarrow \mathcal{C} \models \varphi^n_{\mathcal{A}}$.
- By $\mathcal{A} \simeq^n \mathcal{B}$, we mean that player II has a winning strategy in $\mathrm{EF}_n(\mathcal{A}, \mathcal{B})$.
- **EF theorem**. For all $n \geq 0$, $\mathcal{A} \equiv_n \mathcal{B}$ iff $\mathcal{A} \simeq^n \mathcal{B}$.
- **Corollary** The following are equivalent.
  (1) For any n, there exist $\mathcal{A} \in K$ and $\mathcal{B} \notin K$ such that $\mathcal{A} \equiv_n \mathcal{B}$.
  (2) K is not an elementary class (K cannot be defined by a first-order formula).
- By the EF theorem, DLO is decidable.
- DLO is PSPACE-complete. TQBF is polynomial-time reducible to DLO.
- (Gurevich) For any $m > 0$, for any two finite linear sequences L_1, L_2 of length 2^m or greater, $L_1 \equiv_m L_2$.
- For finite linear orders, there is no first-order formula expressing the parity of its length.
- The connectivity of a graph cannot be defined by a first-order formula.

Thank you for your attention!