Recap
Recursive functions
Programming language TPL
Enumeration theorem
Computably enumerable set
Computable set

Logic and Computation I

Chapter 1 Introduction to theory of computation

Kazuyuki Tanaka

BIMSA

September 24, 2024

Logic and Computation I

- **Part 1. Introduction to Theory of Computation**
- **Part 2. Propositional Logic and Computational Complexity**
- **Part 3. First Order Logic and Decision Problems**
- **Part 4. Modal logic**

Part 1. Schedule

- Sep.10, (1) Automata and monoids
- Sep.12, (2) Turing machines
- Sep.19, (3) Computable functions and primitive recursive functions
- Sep.24, (4) Decidability and undecidability
- Sep.26, (5) Partial recursive functions and computable enumerable sets
- Oct. 8, (6) Rice's theorem and many-one reducibility

Recap
Recursive functions
Programming language TPL
Enumeration theorem
Computably enumerable set
Computable set

Recap

- f is **computable** iff there is a TM such that for an input word $1^{m_1}0\cdots01^{m_k}$, it enters an final state with an remaining sequence $1^{f(m_1,\ldots,m_k)}$ on the tape iff $\{1^{m_1}0\cdots01^{m_k}01^{f(m_1,\ldots,m_k)} : m_1,\ldots,m_k \in \mathbb{N}\}$ is a type-0 language on $\{0,1\}$.
- The **primitive recursive functions** are obtained from constant 0, **successor function** $\mathrm{S}(x) = x+1$, **projection** $\mathrm{P}^n_i(x_1,x_2,\ldots,x_n) = x_i$ $(1 \le i \le n)$, by way of **Composition** and **Primitive recursion**:
If g, h are prim. rec. functions, so is f defined by:

$$f(x_1,\ldots,x_n,0) = g(x_1,\ldots,x_n),$$
$$f(x_1,\ldots,x_n,y+1) = h(x_1,\ldots,x_n,y,f(x_1,\ldots,x_n,y)).$$

- An n-ary relation $R \subset \mathbb{N}^n$ is primitive recursive, if its characteristic function $\chi_R : \mathbb{N}^n \to \{0,1\}$ is primitive recursive.
- The graph of a primitive recursive function is primitive recursive.

Examples

$x+y$, $x \dot{-} y$, $x \cdot y$, x/y, x^y, $x!$, $\max\{x,y\}$, $\min\{x,y\}$ are prim. rec. functions.
$x<y$, $x=y$, $\mathrm{prime}(x)$ are prim. rec. relations.

Example 14

Let $p(x) =$ "$(x+1)$-th prime number ", that is ,

$$p(0)=2, p(1)=3, p(2)=5, \dots$$

Then, $p(x)$ is a primitive recursive function since it is defined as follows.

$$p(0)=2, \quad p(x+1) = \mu y < p(x)!+2\ (p(x)<y \wedge \mathrm{prime}(y)).$$

A finite sequence of natural numbers $(x_0, \dots, x_{n-1})$ can be represented by a unique natural number x, called a sequence number, defined as follows,

$$x = p(0)^{x_0+1} \cdot p(1)^{x_1+1} \cdot \cdots \cdot p(n-1)^{x_{n-1}+1}.$$

Example 15

- Fixing n, a mapping from $(x_0, \ldots, x_{n-1}) \in \mathbb{N}^n$ to its sequence number $x \in \mathbb{N}$ is a primitive recursive function.
- Conversely, let $c(x, i)$ be a function taking the i-th element x_i from x. It is primitive recursive, since

$$x_i = c(x, i) = \mu y < x\ (\neg \exists z < x\, (p(i)^{y+2} \cdot z = x)).$$

- The length of a sequence x, denote $\mathrm{leng}(x)$, is primitive recursive, since

$$\mathrm{leng}(x) = \mu i < x\ (\neg \exists z < x\, (p(i) \cdot z = x)).$$

- Finally, we define a relation $\mathrm{Seq}(x)$ to mean that x is a sequence number. Then it is primitive recursive, since

$$\mathrm{Seq}(x) \Leftrightarrow \forall i < x \forall z < x\ (p(i) \cdot z = x \to i \leq \mathrm{leng}(x)).$$

Recap
Recursive functions
Programming language TPL
Enumeration theorem
Computably enumerable set
Computable set

Gödel numbers

Definition 1.27

Let Ω be a finite (or countably infinite) set of symbols with an injection $\phi : \Omega \to \mathbb{N}$. For a string $s = a_0 \cdots a_{n-1}$ from Ω, the number sequence of $(\phi(a_0) \cdots \phi(a_{n-1}))$, i.e.,

$$p(0)^{\phi(a_0)+1} \cdot p(1)^{\phi(a_1)+1} \cdot \cdots \cdot p(n-1)^{\phi(a_{n-1})+1}$$

is called the **Gödel number** of s, denoted by $\ulcorner s \urcorner$.

The mapping $\ulcorner \quad \urcorner$ is an injection from the set of all strings Ω^* to $\mathbb{N}$.

Example 16

Let $\Omega = \{0, 1, +, (,)\}$, $\phi(0) = 0$, $\phi(1) = 1$, $\phi(+) = 3$, $\phi(( ) = 5$ and $\phi( )) = 6$. Then,

$$\ulcorner (1+0)+1 \urcorner = 2^6 \cdot 3^2 \cdot 5^4 \cdot 7^1 \cdot 11^7 \cdot 13^4 \cdot 17^2$$

Exercise 1.3.3

The symbol set Ω is the same as the example above. "Terms" are defined as below

(1) 0, 1 are terms.

(2) if s and t are terms, so is $(s + t)$.
e.g., $((1 + 0) + 1)$ is a term, but $(1 + 0) + 1$ is not a term.

Show that the predicate $\mathrm{Term}(x)$ expressing "x is the Gödel number of a term" is primitive recursive.

Recursive functions

Definition 1.28

The **recursive functions** are defined as follows

1. **Constant** 0, **Successor** $\mathrm{S}(x) = x + 1$, **Projections** $\mathrm{P}^n_i(x_1, x_2, \cdots, x_n) = x_i$ are recursive functions.
2. **Composition**.
3. **Primitive recursion**.
4. **minimalization** (or **minimization**).
   Let $g : \mathbb{N}^{n+1} \to \mathbb{N}$ be a recursive function such that $\forall x_1 \cdots \forall x_n \exists y\ g(x_1, \cdots, x_n, y) = 0$. Define a function $f : \mathbb{N}^n \to \mathbb{N}$ by

$$f(x_1, \cdots, x_n) = \mu y(g(x_1, \cdots, x_n, y) = 0),$$

where $\mu y(g(x_1, \cdots, x_n, y) = 0)$ denotes the smallest y such that $g(x_1, \cdots, x_n, y) = 0$. Then, f is recursive.

- Recursive functions are (total) computable functions, like primitive recursive functions.
- However, condition 4 in the above definition (not included in the definition of primitive recursive functions) is problematic sometimes, since it is often difficult to guarantee its totality condition $\forall x_1 \cdots \forall x_n \exists y \; g(x_1, \cdots, x_n, y) = 0$ in a absolutely computable way, or in a rigid formal system.
- For instance, the class of recursive functions allowed in Peano arithmetic does not match the class of recursive functions allowed in ZF set theory.
- A function defined by removing this totality condition is called **a partial recursive function**, and we will discuss it later (in Lecture 5).

§1.4. Computability and Incomputability

- Today, we will only consider a deterministic single-tape Turing machine on $\Omega = \{0, 1, \mathrm{B}\}$.
- We will introduce a **P**rogramming **L**anguage, called **TPL**, that has an instruction for each operation of **T**uring machine.
- Any Turing machine can be emulated by a TPL program on a unique Turing machine (called a universal Turing machine).
- Finally, we will prove the existence of an incomputable (non-computable) set K.

Definition 1.29 (Programming language TPL)

Instructions (code $\sharp$, the corresponding TM operations)

halt	(code 0, enter a final state)
moveright	(code 1, move the head to right by one cell)
moveleft	(code 2, move the head to left by one cell)
write 0	(code 3, write "0" on the tape)
write 1	(code 4, write "1" on the tape)
write B	(code 5, write "B" On the tape)
goto l	(code $6+3l$, jump to the l-th instruction)
if 0 **then goto** l	(code $7+3l$, if TM reads 0, jump to the l-th instruction)
if 1 **then goto** l	(code $8+3l$, if TM reads 1, jump to the l-th instruction)

Definition 1.29 (continued)

A **program** of TPL is a list of instructions separated by ";".

For readability, a line number is added at each instruction.
In the instruction "**goto** l", l corresponds to such a line number.

An example of TPL program $\mathcal{P}_0$

0: **if** 1 **then goto** 2;
1: **goto** 1;
2: **moveright**;
3: **if** 1 **then goto** 1;
4: **if** 0 **then goto** 6;
5: **halt**;
6: **moveright**;
7: **goto** 0

The left program intends to accept the language $1(01)^*$

Definition 1.30 (TM $\mathcal{M}_{\mathcal{P}}$ realizes TPL program $\mathcal{P}$)

Let $\mathcal{P}$ be be a TPL program. We define a (deterministic) Turing machine $\mathcal{M}_{\mathcal{P}} = (Q, \Omega, \delta, q_0, F)$ which realizes $\mathcal{P}$. Here, $Q = \{0, 1, \dots, n-1\}$ is the set of line numbers of $\mathcal{P}$. $\Omega = \{0, 1, \mathrm{B}\}$. $q_0 = 0$, $F = \{$a line number of **halt**$\}$. The transition function $\delta : Q \times \Omega \to \Omega \times \{L, R, N\} \times Q$ is defined as follows.

l: **halt**,	$\delta(l, x) = (x, N, l)$,
l: **moveright**,	$\delta(l, x) = (x, R, l+1)$,
l: **moveleft**,	$\delta(l, x) = (x, L, l+1)$,
l: **write** $?$,	$\delta(l, x) = (?, N, l+1)$, for $? = 0, 1, \mathrm{B}$,
l: **goto** k,	$\delta(l, x) = (x, N, k)$,
l: **if** $?$ **then goto** k,	$\delta(l, ?) = (?, N, k)$ and
	$\delta(l, y) = (y, N, l+1)$ for $y \neq ?$.

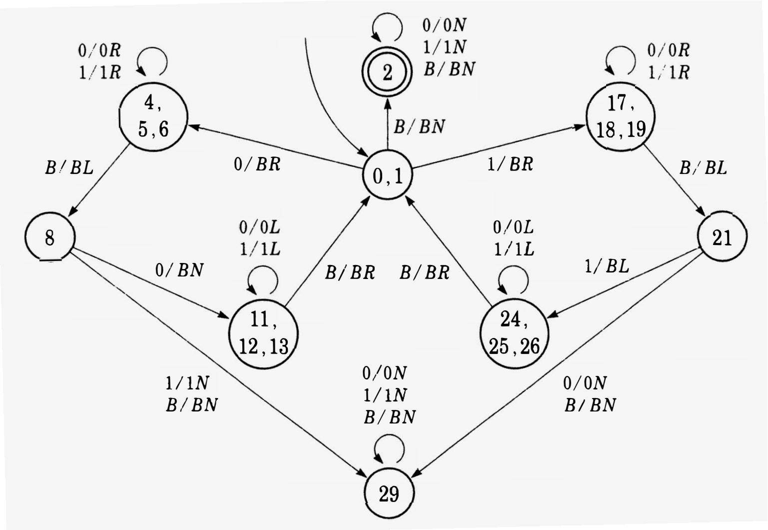
The language accepted by TPL $\mathcal{P}$ is the language accepted by the associated Turing machine $\mathcal{M}_{\mathcal{P}}$. The partial function $f : \Omega^* \to \Omega^*$ defined by $\mathcal{P}$ is a function defined by $\mathcal{M}_{\mathcal{P}}$.

Program $\mathcal{P}_0 \Rightarrow$ TM $\mathcal{M}_{\mathcal{P}_0}$

We define a (deterministic) Turing machine $\mathcal{M}_{\mathcal{P}_0} = (Q, \Omega, \delta, q_0, F)$, where $Q = \{0, 1, \ldots, 7\}$, $\Omega = \{0, 1, \mathrm{B}\}$, $q_0 = 0$, $F = \{5\}$, and
δ is defined as follows: for any $x \in \Omega$,

0: **if** 1 **then goto** 2;	$\delta(0,1) = (1, N, 2)$, $\delta(0, y) = (y, N, 1)$ for $y \neq 1$
1: **goto** 1;	$\delta(1, x) = (x, N, 1)$
2: **moveright**;	$\delta(2, x) = (x, R, 3)$
3: **if** 1 **then goto** 1;	$\delta(3,1) = (1, N, 1)$, $\delta(3, y) = (y, N, 4)$ for $y \neq B$
4: **if** 0 **then goto** 6;	$\delta(4,0) = (0, N, 6)$, $\delta(4, y) = (y, N, 5)$ for $y \neq 1$
5: **halt**;	$\delta(5, x) = (x, N, 5)$
6: **moveright**;	$\delta(6, x) = (x, R, 7)$
7: **goto** 0	$\delta(7, x) = (x, N, 0)$

Example 17. TM $\mathcal{M}_1$ such that $L(\mathcal{M}_1) = \{ww^R : w \in \{0,1\}^*\}$



A TPL program $\mathcal{P}_1$ for $L(\mathcal{M}_1)$

0: **if** 0 **then goto** 3;
1: **if** 1 **then goto** 16;
2: **halt**;

3: **write** B;
4: **moveright**;
5: **if** 0 **then goto** 4;
6: **if** 1 **then goto** 4;
7: **moveleft**;
8: **if** 0 **goto** 10;
9: **goto** 29;
10: **write** B;
11: **moveleft**;
12: **if** 0 **then goto** 11;
13: **if** 1 **then goto** 11;
14: **moveright**;
15: **goto** 0;

16: **write** B;
17: **moveright**;
18: **if** 0 **then goto** 17;
19: **if** 1 **then goto** 17;
20: **moveleft**;
21: **if** 1 **goto** 23;
22: **goto** 29;
23: **write** B;
24: **moveleft**;
25: **if** 0 **then goto** 24;
26: **if** 1 **then goto** 24;
27: **moveright**;
28: **goto** 0;

29: **goto** 29

- One step of TM $\mathcal{M}_1$ is described as two or three instructions in $\mathcal{P}_1$.
- For instance, look at $(4,5,6) \leftarrow (0,1)$ in the figure of Example 17 where the edge is labeled by 0/BR. Then, this step is expressed in $\mathcal{P}_1$ as
  0: **if** 0 **then goto** 3;
  3: **write** B;
  4: **moveright**;
- $\mathcal{P}_1$ was made very efficiently. But without consideration of efficiency, it is routine to make a TPL program for a given TM.

Theorem 1.31

For any Turing machine $\mathcal{M}$, there exists a TPL program $\mathcal{P}$ such that $L(\mathcal{M}) = L(\mathcal{M}_{\mathcal{P}})$.

Proof. (Leave it to the students.)

A program $\mathcal{P}$ is a sequence of instructions with codes $c_0, c_1, \ldots, c_l$.

Example: $\mathcal{P}_0$

0: **if** 1 **then goto** 2;	Code $c_0 = 8 + 3 \cdot 2 = 14$
1: **goto** 1;	Code $c_1 = 6 + 3 \cdot 1 = 9$
2: **moveright**;	Code $c_2 = 1$
$\vdots$	$\vdots$

$\mathcal{P}$ can be represented by a sequence $1^{c_0}0 \cdots 01^{c_l}$ on $\{0,1\}^*$.

The **Gödel number** of a program $\ulcorner \mathcal{P} \urcorner$ is

$$p(0)^{c_0+1} \cdot p(1)^{c_1+1} \cdot \cdots \cdot p(l)^{c_l+1}.$$

According to the previous theorem, for any TM $\mathcal{M}$, there is a TPL program $\mathcal{P}_{\mathcal{M}}$. The Gödel number $\ulcorner \mathcal{P}_{\mathcal{M}} \urcorner$ is called the index (or code) of TM $\mathcal{M}$.

Partial computable functions

- The definition of computable functions in Lecture 3 can be applied to **partial functions**. Namely, a partial function $f : \mathbb{N}^k \longrightarrow \mathbb{N}$ is **computable** if $\{1^{m_1}0 \cdots 01^{m_k}01^{f(m_1,\ldots,m_k)} : m_1, \ldots, m_k \in \mathbb{N}\}$ is a 0-type language.
- Then, the partial function f realized by $\mathcal{M}$ with index e is represented by $\{e\}^k$ (or simply $\{e\}$) (called **Kleene's bracket notation**).
- When e is not a code of TM, $\{e\}$ is regarded as a partial function with empty domain.

Each TPL Program $\mathcal{P}$ is executed as a distinct TM $\mathcal{M}_{\mathcal{P}}$. However, we can construct a "Universal Turing Machine" as a interpreter of TPL Programs. More strictly, we have the following theorem.

Theorem 1.32 (Enumeration theorem)

For any $n \geq 0$, there exists a natural number e_n such that for any d, $x_1, \ldots, x_n$,

$$\{e_n\}^{n+1}(d, x_1, \ldots, x_n) \sim \{d\}^n(x_1, \ldots, x_n).$$

$f(x_1, \ldots, x_n) \sim g(x_1, \ldots, x_n)$ means either both sides are not defined or they are defined with the same value.

This theorem affirms the existence of a universal TM with index e_n that is able to mimic any TM with index d.

Proof. We will construct a universal Turing machine $\mathcal{M}$ with index e_n.

- $\mathcal{M}$ has one input tape and two working tapes.
- Let $1^d 0 1^{x_1} 0 \cdots 0 1^{x_n}$ be an input on the first tape.
- Let the index part 1^d represent the program
  $\{$the instruction of code c_0;
  the instruction of code c_1;
  $\cdots$;
  the instruction of code $c_l\}$.
- Write $1^{c_0} 0 \cdots 0 1^{c_l}$ on the 2nd tape and remove $1^d 0$ on the 1st tape.
- Execute the instructions on the 2nd tape sequentially, rewriting the string on the 1st tape with the help of the 3rd tape.

Proof.(Continued)

- The 3rd tape will be used to find the next executable operation when the **goto** instruction or **if** ? **then goto** instruction is executed on the 2nd tape.
  - For instance, 1^{6+3l} sandwiched between two 0's means **goto** l, and so the next executable instruction is given by the sequence of 1's between the l-th 0 and the $l+1$-th 0. To find it on the 2nd tape, we need to store the number of 0's counted from the left to the end of the string.
- Instructions other than **goto** and **if** ? **then goto** can be easily executed, and finding the next executable instruction is also obvious.
- When **halt** instruction is executed, $\mathcal{M}$ enters a final state.
- At that time, $1^{\{d\}^n(x_1,\ldots,x_n)}$ is written on the 1st tape.

$\square$

Definition 1.33

A set $X \subset \mathbb{N}^n$ is called computably enumerable, CE for short, if

$$\{1^{x_1}0 \cdots 01^{x_n} : (x_1, \ldots, x_n) \in X\}$$

is a type-0 language.

In other words, $X \subset \mathbb{N}^n$ is CE iff it is the domain of some partial computable function. Other equivalent definitions will be given in the next lecture.

Now, let K be a set of natural numbers defined as

$$\mathrm{K} := \{e : e \in \mathrm{dom}(\{e\}^1)\} = \{e : (e, e) \in \mathrm{dom}(\{e_1\}^2)\}.$$

where e_1 is the code of the universal TM in the previous theorem. We call K the **halting problem**. Strictly speaking, this is a special kind of halting problem, and the general case K_0 will be given later.

Theorem 1.34 (Turing)

K is CE but its complement $\mathbb{N} - \mathrm{K}$ is not CE.

Proof.

- To show K is CE.
  We construct a TM $\mathcal{M}$ accepting $\{1^e : e \in \mathrm{K}\}$ as follows.
  - For input 1^e, it rewrites as 1^e01^e on the tape.
  - Then, $\mathcal{M}$ mimics the universal TM that realizes $\{e_1\}^2$.
  - This TM enters a final state, if $(e, e) \in \mathrm{dom}(\{e_1\}^2)$, i.e., $e \in \mathrm{K}$.
- By contradiction, assume that $\mathbb{N} - \mathrm{K}$ is a CE set.
  Assume a TM with code d that accepts $\{1^e : e \notin K\}$. At this time,

$$d \in \mathrm{K} \Leftrightarrow d \in \mathrm{dom}(\{d\}^1) \Leftrightarrow d \in \{e : e \notin \mathrm{K}\} \Leftrightarrow d \notin \mathrm{K}$$

Therefore, either $d \in \mathrm{K}$ or $d \notin \mathrm{K}$ leads to a contradiction.

Definition 1.35

A set $X \subset \mathbb{N}^n$ is **computable** (or **recursive**, **decidable**) if both X and its complement are CE.

- K is an incomputable CE set.
- A set $X \subset \mathbb{N}^n$ is computable iff its characteristic function χ_R is computable.
  - $(\Rightarrow)$ If we have partial computable functions f and g with $\mathrm{dom}(f) = X$ and $\mathrm{dom}(g) = \mathbb{N}^n - X$, then for any input $1^{x_1}0 \cdots 01^{x_n}$, execute the computations for f, g in parallel and decide the output (1 or 0) depending on which one stops first. Such computation always terminates.
- If a function $f(x)$ has a finite value at $x = n$, we write $f(n)\downarrow$. That is

$$f(n)\downarrow \Leftrightarrow n \in \mathrm{dom}(f).$$

Then we also write K as

$$\mathrm{K} = \{e : \{e\}(e)\downarrow\}$$

Exercise 1.4.1

Show that the following two sets are incomputable CE set.

$$\mathrm{K}_0 = \{(x, e) : \{e\}(x)\downarrow\},$$

$$\mathrm{K}_1 = \{e : \mathrm{dom}(\{e\}) \neq \emptyset\}.$$

- K_0 is the original **halting problem**: given a program and input, decide when the machine will halt.
- Since we use the special halting problem K more frequently, we refer K_0 as the "membership decision problem (MEM)".

Summary

- Enumeration theorem: Any Turing machine can be emulated by a TPL program on a universal Turing machine.
- A set $X \subset \mathbb{N}^n$ is CE if $\{1^{x_1}0\cdots01^{x_n} : (x_1, \ldots, x_n) \in X\}$ is a type-0 language.
- X is computable if both X and X^c are CE.
- K is CE but not computable.

Further readings

N. Cutland. *Computability: An Introduction to Recursive Function Theory*, Cambridge University Press, 1st edition, 1980.

Recap
Recursive functions
Programming language TPL
Enumeration theorem
Computably enumerable set
Computable set

Thank you for your attention!