Logic and Foundation I

Part 2. First-order logic

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Logic and Foundations I

- Part 1. Equational theory
- Part 2. First order theory
- **Part 3. Model theory**
- Part 4. First order arithmetic and incompleteness theorems

Part 2. Schedule

- Nov. 16, (1) $\forall$-theory and $\forall\exists$-theory
- Nov. 23, (2) Horn theory and reduced products
- Nov. 30, (3) Ultra products and non-standard analysis

Today's topics

1. Recap
2. Ultraproducts
3. Homework
4. Non-standard analysis

Recap

Definition

Let I be a non-empty set. $\mathcal{F} \subseteq \mathcal{P}(I)$ is said to be **filter** on I if the following are satisfied.

(1) $\varnothing \notin \mathcal{F}$, $I \in \mathcal{F}$.

(2) $X \in \mathcal{F}, X \subseteq Y \subseteq I \Rightarrow Y \in \mathcal{F}$.

(3) $X, Y \in \mathcal{F} \rightarrow X \cap Y \in \mathcal{F}$.

Let I be an infinite set.

1. The collection of co-finite subsets of I is a filter, called a **Fréchet filter**.
2. For each $i \in I$, $\{X \subseteq I : i \in X\}$ is a filter, called a **principal filter**.

Lemma

If $S \subset \mathcal{P}(I)$ has the **finite intersection property***: for any finite subset $\{J_1, \dots J_n\} \subset S$,*

$$J_1 \cap \cdots \cap J_n \neq \varnothing,$$

then there exists a filter $\mathcal{F}$ including S.

Definition

Let $\mathfrak{A}_i = (A_i, \mathtt{f}^{\mathfrak{A}_i}, \dots, \mathrm{R}^{\mathfrak{A}_i}, \dots)\,(i \in I)$ be an $\mathcal{L}$-structure.
Let $\mathcal{F}$ be a filter on I. Then, we define the binary relation $\approx_{\mathcal{F}}$ on $\prod A_i$ as follows

$$a \approx_{\mathcal{F}} b \quad \Leftrightarrow \quad \{i \in I : a(i) = b(i)\} \in \mathcal{F}.$$

Lemma

$\approx_{\mathcal{F}}$ is a congruence relation.

Thus, we can define the quotient structure in the same way as for the algebraic structure.

Definition

Let $\mathfrak{A}_i = (A_i, \mathtt{f}^{\mathfrak{A}_i}, \dots, \mathrm{R}^{\mathfrak{A}_i}, \dots)\,(i \in I)$ be $\mathcal{L}$-structures. Let $\mathcal{F}$ be a filter on I. Then, the following $\mathcal{L}$-structure is called the **reduced product** of $\mathfrak{A}_i$, denoted by $\prod \mathfrak{A}_i/\mathcal{F}$.

$$\left(\prod A_i/\mathcal{F}, \mathtt{f}^{\prod \mathfrak{A}_i/\mathcal{F}}, \dots, \mathrm{R}^{\prod \mathfrak{A}_i/\mathcal{F}}, \dots\right)$$

- $\theta_0 \vee \neg\theta_1 \vee \cdots \vee \neg\theta_n$ and $\neg\theta_1 \vee \cdots \vee \neg\theta_n$ are called **basic Horn formulas**, if $\theta_i \; (i < n)$ are atomic formulas.

- A formula constructed from the basic Horn formulas by using only $\wedge$, $\forall$, and $\exists$ is called a **Horn formula**. A set of Horn sentences is called a **Horn theory**.

For $a_1, \ldots, a_n \in \prod A_i$, we set $\|\varphi(a_1, \ldots, a_n)\| := \{i \in I : \mathfrak{A}_i \models \varphi(a_1(i), \ldots, a_n(i))\}$.

Lemma

Let $\varphi(x_1, \ldots, x_n)$ be a Horn formula, then for $a_1, \ldots, a_n \in \prod A_i$,

$$\|\varphi(a_1, \ldots, a_n)\| \in \mathcal{F} \Rightarrow \prod \mathfrak{A}_i/\mathcal{F} \models \varphi([a_1], \ldots, [a_n]).$$

Theorem (Keisler-Galvin)

The following are equivalent:

(1) $\mathrm{Mod(T)}$ *is closed under reduced products.*

(2) *There exists a Horn theory T' such that* $\mathrm{Mod(T)} = \mathrm{Mod(T')}$.

A proof (1) ⇒ (2) can be found in Chang-Keisler's classic textbook *Model Theory*.

- A sentence with several $\forall$ in front of a basic Horn formula is called a **$\forall$-Horn sentence** (or simply called a Horn sentence in some literature). A collection of such sentences is called a **$\forall$-Horn theory** (or simply a Horn theory).
- A $\forall$-Horn theory is a nice extension of equational theory. The following theorem is a counter part of Birkhoff's equational class theorem. It can be proven similarly, and we leave the details to the reader.

Theorem

Let $\mathcal{K}$ be a class of $\mathcal{L}$-structures, then the following are equivalent:

(1) *$\mathcal{K}$ is closed under direct products, substructures, and isomorphic images.*

(2) *$\mathcal{K}$ is closed under reduced products, substructures, and isomorphic images.*

(3) *There exists a $\forall$-Horn theory T such that $\mathrm{Mod(T)} = \mathcal{K}$.*

Ultraproducts

In the following, we will consider the necessary and sufficient conditions for a class of structures to be axiomatized by first order logic, that is, be expressed as $\mathrm{Mod}(\mathrm{T})$.

Definition

A class $\mathcal{K}$ of $\mathcal{L}$-structures is called an **elementary class** if there exists a set T of sentences such that $\mathcal{K} = \mathrm{Mod}(\mathrm{T})$. In this case, we write

$$\mathcal{K} \in \mathrm{EC}_{\Delta}.$$

To characterize elementary classes, we use a kind of reduced product called "ultraproduct". To define it, we first introduce an ultrafilter.

Definition

The filter $\mathcal{F}$ on I is an **ultrafilter** (maximal filter) if the following properties are satisfied.

$$\forall X \subset I(X \in \mathcal{F} \vee I - X \in \mathcal{F}).$$

Lemma

Every filter $\mathcal{F}$ can be expanded to an ultrafilter $\mathcal{U}$.

Proof. Consider the class of all filters including a given filter $\mathcal{F}$.
Since it is closed under the union of chains, by Zorn's lemma, there is a maximal filter $\mathcal{U}$ which is an ultrafilter. □

A principal filter is an ultrafilter. There exists an ultrafilter which is **non-principal**.

Lemma

There exists a non-principal ultrafilter $\mathcal{U}$ on any infinite set I.

Proof.
Let I be an infinite set, and $\mathcal{F}$ be a Fréchet filter on it (a subset of I whose complement is finite). By the above lemma, an ultrafilter $\mathcal{U}$ can be obtained by expanding $\mathcal{F}$. Then $\mathcal{U}$ is non-principal, since for each $i \in I$, $I - \{i\} \in \mathcal{F} \subseteq \mathcal{U}$, so we have $\{i\} \notin \mathcal{U}$.

Stone's representation theorem

We prove Stone's representation theorem using an ultrafilter.

Theorem (Stone's representation theorem)

For any Boolean algebra $\mathfrak{B}$, there exists a set X, and $\mathfrak{B}$ can be embedded in the power set algebra $\mathfrak{P}(X)$.
In particular, if $\mathfrak{B}$ is finite, it is isomorphic to $\mathfrak{P}(X)$.

Proof.

- Let $\mathfrak{B} = (B, \vee, \wedge, \neg, 0, 1)$ be a Boolean algebra. Filters, Ultrafilters, and others can naturally be defined for a subset $F \subseteq B$ with the ordering $x \leq y \Leftrightarrow x \wedge y = x$.
  Let X be the set of all ultrafilters of B and $\mathcal{P}(X)$ be its power set.
- Define $f : B \to \mathcal{P}(X)$ as follows: $f(b)$ is the set of ultrafilters containing b.
  Then, $f : B \to \mathcal{P}(X)$ is embedding.
- If $\mathfrak{B}$ is finite, any ultrafilter must be a principal filter. And its generator is an atom (non-zero minimal element) in $\mathfrak{B}$. So, let X be the set of atoms. It is easy to see that $\mathfrak{B}$ and $\mathfrak{P}(X)$ are isomorphic. □

Ultraproducts

Definition (Ultraproduct)

The reduced product $\prod \mathfrak{A}_i/\mathcal{U}$ for an ultrafilter $\mathcal{U}$ is called an **ultraproduct**.

Theorem (Łos)

Let $\mathcal{U}$ be an ultrafilter. For any formula $\varphi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in \prod A_i$,
$\prod \mathfrak{A}_i/\mathcal{U} \models \varphi([a_1], \ldots, [a_n]) \Leftrightarrow \|\varphi(a_1, \ldots, a_n)\| \in \mathcal{U}$.

Proof. By induction on the construction of formulas. The atomic formulas and formulas beginning with $\wedge$ and $\exists$ can be treated in the same way as reduced products.
Then we only need to treat the case of negation $\neg\varphi$.

$$\begin{aligned}\prod \mathfrak{A}_i/\mathcal{U} \models \neg\varphi &\Leftrightarrow \prod \mathfrak{A}_i/\mathcal{U} \not\models \varphi \\ &\Leftrightarrow \|\varphi\| \notin \mathcal{U} \quad (\because \text{induction hypothesis}) \\ &\Leftrightarrow \|\neg\varphi\| \in \mathcal{U} \quad (\because \text{maximality of } \mathcal{U}). \quad \square\end{aligned}$$

Problem 9

Use ultraproducts to show that any field $\mathcal{F}$ has algebraic closure $\overline{\mathcal{F}}$.

Solution:

- We fix a field $\mathcal{F}$ in a language with constants for their elements.
- Let $\mathcal{F}_P$ be a splitting field of a polynomial P, and for each $Q \in \mathcal{F}[X]$, we put

$$J_Q = \{P \in \mathcal{F}[X] : Q \text{ is splitted into linear factors over } \mathcal{F}_P\}.$$

- Then, $\{J_Q : Q \in \mathcal{F}[X] \text{ and } Q \text{ is not a constant.}\}$ has the finite intersection property $(\because Q_1 \cdots Q_n \in J_{Q_1} \cap \cdots \cap J_{Q_n})$. Therefore, it can be expanded to an ultrafilter $\mathcal{U}$.

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- Now consider the ultraproduct $\prod \mathcal{F}_P/\mathcal{U}$, which is a field extension of $\mathcal{F}$.
- For any (non-constant) polynomial $Q \in \mathcal{F}[X]$, the sentence "Q can be splitted over $\mathcal{F}_P$" is true for all P belonging to $J_Q \in \mathcal{U}$, and so it holds in $\prod \mathcal{F}_P/\mathcal{U}$.
- Therefore, $\prod \mathcal{F}_P/\mathcal{U}$ is an algebraically closed field.
- Finally, we define $\overline{\mathcal{F}}$ to be the set of elements of $\prod \mathcal{F}_P/\mathcal{U}$ which is a root of some $P \in \mathcal{F}[X]$. Clearly, $\overline{\mathcal{F}}$ is an algebraic extension of $\mathcal{F}$.
- Now, suppose for the contrary that there is a polynomial in $\overline{\mathcal{F}}[X]$ that has no root in $\overline{\mathcal{F}}$. Then, the root should be to expressed as a root of the polynomial of $\mathcal{F}$ ("Algebraic extension" is transitive), which contradicts with the definition of $\overline{\mathcal{F}}$.
- Therefore, $\overline{\mathcal{F}}$ is an algebraic closure of $\mathcal{F}$.

Theorem (Frayne-Morel-Scott)

A class of structures $\mathcal{K}$ is an elementary class (EC_Δ) iff it is closed under elementary equivalences and ultraproduct.

Proof.

- ($\Rightarrow$) is clear. To show ($\Leftarrow$), suppose that $\mathcal{K}$ is closed under elementary equivalences and utraproducts. Let $T = \{\sigma : \forall \mathfrak{A} \in \mathcal{K}, \mathfrak{A} \models \sigma\}$ and we claim $\mathcal{K} = \mathrm{Mod}(T)$. $\mathcal{K} \subseteq \mathrm{Mod}(T)$ is clear. To show $\mathrm{Mod}(T) \subseteq \mathcal{K}$, we take any $\mathfrak{B} \in \mathrm{Mod}(T)$. Let I be the set of finite subsets of $\mathrm{Th}(\mathfrak{B})$.
- By way of contradiction, assume there is an $i \in I$ such that $\forall \mathfrak{A} \in \mathcal{K}\,(\mathfrak{A} \not\models i)$. Suppose $i = \{\varphi_1, \ldots, \varphi_n\}$. Since for any $\mathfrak{A} \in \mathcal{K}$, $\mathfrak{A} \models \neg\varphi_1 \vee \cdots \vee \neg\varphi_n$, we have $\neg\varphi_1 \vee \cdots \vee \neg\varphi_n \in T$. Since $\mathfrak{B} \models T$, we have $\mathfrak{B} \models \neg\varphi_k$ for some $k \in i$, which contradicts $\varphi_k \in i \subseteq \mathrm{Th}(\mathfrak{B})$. Therefore, for any $i \in I$, there exists $\mathfrak{A}_i \in \mathcal{K}$ such that $\mathfrak{A}_i \models i$.
- We can construct a model $\mathfrak{A}$ of $T = \mathrm{Th}(\mathfrak{B})$ by ultraproduct as in the proof of compactness theorem. Then since $\mathcal{K}$ is closed under ultraproducts, we have $\mathfrak{A} \in \mathcal{K}$. Moreover, because $\mathcal{K}$ is closed under elementary equivalence, $\mathfrak{A} \equiv \mathfrak{B}$ implies $\mathfrak{B} \in \mathcal{K}$. □

Definition

$\prod \mathfrak{A}_i/\mathcal{U}$ is called an **ultrapower** of $\mathfrak{A}$, denoted by $\mathfrak{A}^I/\mathcal{U}$, if $\mathfrak{A}_i = \mathfrak{A}$ for each $i \in I$.

Let $\lambda i.a$ denote a function which always takes the value a. For $a \in |\mathfrak{A}|$, we put

$$^*a = [\lambda i.a] \in |\mathfrak{A}^I/\mathcal{U}|$$

and define a function $d : |\mathfrak{A}| \to |\mathfrak{A}^I/\mathcal{U}|$ by $d(a) = {}^*a$, which is called a **canonical embedding**.

Definition

An embedding $\phi : \mathfrak{A} \to \mathfrak{B}$ is said to be **elementary** if $\phi(\mathfrak{A}) \prec \mathfrak{B}$.

Theorem

Let $\prod \mathfrak{A}^I/\mathcal{U}$ be an ultrapower of $\mathfrak{A}$. Then the canonical embedding $d : |\mathfrak{A}| \to |\mathfrak{A}^I/\mathcal{U}|$ is elementary. In particular, $\mathfrak{A} \equiv \mathfrak{A}^I/\mathcal{U}$.

Proof. For any formula $\varphi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in |\mathfrak{A}|$, by Łos theorem,

$$\begin{aligned}\mathfrak{A}^I/\mathcal{U} \models \varphi({}^*a_1, \ldots, {}^*a_n) &\Leftrightarrow \{i \in I : \mathfrak{A} \models \varphi(a_1, \ldots, a_n)\} \in \mathcal{U} \\ &\Leftrightarrow \mathfrak{A} \models \varphi(a_1, \ldots, a_n).\end{aligned}$$

Thus, d is an elementary embedding. Since $d(\mathfrak{A}) \cong \mathfrak{A}$, $\mathfrak{A} \equiv \mathfrak{A}^I/\mathcal{U}$. □

Theorem (Keisler-Shelah)

$\mathfrak{A} \equiv \mathfrak{B} \Leftrightarrow$ *There exist an I and a ultrafilter $\mathcal{U}$ such that $\mathfrak{A}^I/\mathcal{U} \cong \mathfrak{B}^I/\mathcal{U}$.*

Proof. ($\Leftarrow$) is derived from the last theorem. The proof of ($\Rightarrow$) is omitted since it is too technically involved. See *Model Theory: Third Edition - C.C. Chang, H. Jerome Keisler* for details. □

Assuming the Keisler-Shelah theorem, we obtain the following.

Corollary

The structural class $\mathcal{K}$ is the elementary class (EC_Δ) iff the following two conditions hold.

(1) *$\mathcal{K}$ is closed under ultraproducts and isomorphisms.*

(2) *$\mathfrak{A}^I/\mathcal{U} \in \mathcal{K} \Rightarrow \mathfrak{A} \in \mathcal{K}$. (It is closed under inverses of ultrapower).*

Proof. To show the sufficient condition, we prove $\mathcal{K}$ is closed by elementary equivalence.

- Let $\mathfrak{A} \equiv \mathfrak{B}$ and $\mathfrak{A} \in \mathcal{K}$.
- By the Keisler-Shelah theorem, there is an ultrapower $\mathcal{U}$ such that $\mathfrak{A}^I/\mathcal{U} \cong \mathfrak{B}^I/\mathcal{U}$.
- Since $\mathcal{K}$ is closed under ultraproduct, $\mathfrak{A}^I/\mathcal{U} \in \mathcal{K}$.
- Because $\mathcal{K}$ is closed under isomorphisms, $\mathfrak{B}^I/\mathcal{U} \in \mathcal{K}$.
- Moreover, by condition (2), we have $\mathfrak{B} \in \mathcal{K}$. □

- A class $\mathcal{K}$ of structures in a language $\mathcal{L}$ is called a **projective class** or **pseudo-elementary class**, denoted $\mathcal{K} \in \mathrm{PC}_\Delta$, if there exists an elementary class $\mathcal{K}' \in \mathrm{EC}_\Delta$ in an extended language $\mathcal{L}' \supseteq \mathcal{L}$ such that

$$\mathcal{K} = \{\mathfrak{A} : \mathfrak{A} \text{ is a reduct of a model in } \mathcal{K}' \text{ to } \mathcal{L}\}.$$

- For example, the class of orderable groups is a projective class.
- It is easy to see that PC_Δ is also closed under ultraproducts and isomorphisms. Various characterizations are also known for PC_Δ.
- The following one is particularly interesting, and so important as it allows us to derive Craig's interpolation theorem.

Theorem

If $\mathcal{K}, \mathcal{K}' \in \mathrm{PC}_\Delta$ and $\mathcal{K} \cap \mathcal{K}' = \varnothing$, then there exists $\mathcal{J} \in \mathrm{EC}$ such that $\mathcal{K} \subseteq \mathcal{J}$ and $\mathcal{J} \cap \mathcal{K}' = \varnothing$ where $\mathcal{J} \in \mathrm{EC}$ means that $\mathcal{J} = \mathrm{Mod}(\{\sigma\})$ with a single sentence σ.

Homework Problem

Show that $\mathcal{K}$ is finitely axiomatizable iff both $\mathcal{K}$ and its complement are closed under ultraproducts and elementary equivalence.

Non-standard analysis

- Using ultrapowers, we can construct a large non-standard structure that properly includes a common standard structure such as natural numbers, real numbers, and function spaces as elementary substructures.
- In particular, a non-standard model of real numbers includes infinities and infinitesimals as elements, and thus provides the first rational model for Leibniz's style of infinitesimal analysis.
- Non-standard methods have been applied to various fields of mathematics. In particular, its application to analysis is called **non-standard analysis**.
- From now on, we fix a non-principal ultrafilter $\mathcal{U}$ on the natural numbers $\omega\,(=\mathbb{N})$ and denote the ultrapower $\prod \mathfrak{A}^I/\mathcal{U}$ of a structure $\mathfrak{A}$ by ${}^*\mathfrak{A}$.
- As shown before, there is a natural embedding $d(a) = {}^*a$ from $\mathfrak{A}$ to ${}^*\mathfrak{A}$. Identifying $\mathfrak{A}$ and its image $d(\mathfrak{A})$, $\mathfrak{A}$ can be regarded as an elementary substructure of ${}^*\mathfrak{A}$.
- The structures like $\mathfrak{N} = (\mathbb{N}, +, \bullet, 0, 1, <)$ and $\mathfrak{R} = (\mathbb{R}, +, \bullet, 0, 1, <)$ etc. are called **standard models**. ${}^*\mathfrak{N}$, ${}^*\mathfrak{R}$, etc. are called their **non-standard models**.

- A standard model and its non-standard counterpart can not be distinguished in terms of elementary (first-order) propositions. But, there might be fundamental properties that cannot be expressed in an elementary manner, e.g., the Archimedean property.
- An ordered field $\mathfrak{A}$ is **Archimedean** if for any positive elements $a, b \in A$ there exists a sufficiently large natural number $n \in \mathbb{N}$ such that $b < a + a + \cdots + a$ (n times).

Theorem

${}^*\mathfrak{R}$ *is a non-Archimedean ordered field.*

Proof.

- Since $\mathfrak{R}$ is an ordered field and such a property can be described in elementary way, ${}^*\mathfrak{R}$ is also an ordered field.

Claim: ${}^*\mathfrak{R}$ is non-Archimedean

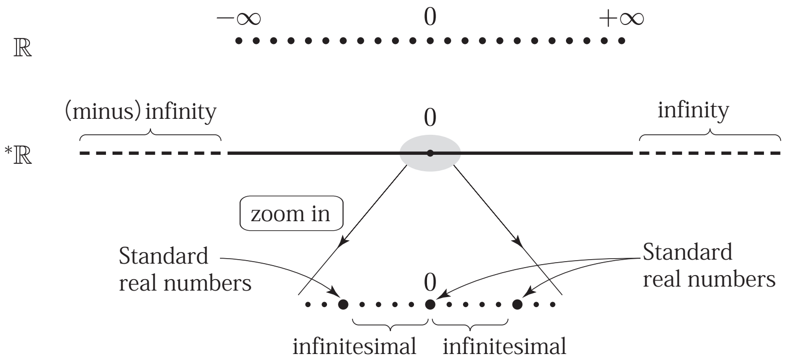
- Let $s = \langle 1, 2, 3, \ldots \rangle \in |\mathfrak{R}^\omega|$ and $N = [s] \in |{}^*\mathfrak{R}|$.
- Then, for any natural number $n \in \mathbb{N}$, we have

$$N > \underbrace{{}^*1 + {}^*1 + \cdots + {}^*1}_{n \text{ times}},$$

since $\{i : s(i) > n\} \in \mathcal{U}$. □

Definition

- An element a of $|{}^*\mathfrak{R}|$ is **infinite** if $\forall b \in \mathbb{R}\ b < |a|$. An element that is not infinite is said to be **finite**.
- The element a of $|{}^*\mathfrak{R}|$ is **infinitesimal** if $\forall b\,(> 0) \in \mathbb{R},\quad |a| < b$.



Example

$N = [\langle 1, 2, 3, \dots \rangle]$ is infinite, $1/N = [\langle 1/1, 1/2, 1/3, \dots \rangle]$ is infinitesimal.

Problem

(1) Show that the set of all infinitesimals is closed under the operations $+$ and $\cdot$.
(2) Show that a is infinite and $1/a$ is infinitesimal.

Definition

For $a, b \in |{}^*\mathfrak{R}|$, $a \approx b \Leftrightarrow a - b$ is infinitesimal.

- It is easy to see that $\approx$ is an equivalence relation and also preserves the operations of $+$ and $\bullet$.

Lemma

For a finite real number $a \in |{}^*\mathfrak{R}|$, there exists a unique $b \in \mathbb{R}$ such that $a \approx b$.

Proof. Set $b = \inf\{x \in \mathbb{R} : a < x\}$. Uniqueness is obvious. □

- Such a b in the above lemma is called the **standard part** of a and is denoted by $\mathrm{st}(a)$. Thus, $a - \mathrm{st}(a)$ is infinitesimal.
- Every finite non-standard real number a can be uniquely represented by the sum of the standard real number $\mathrm{st}(a)$ and an infinitesimal.

Lemma

If $s = \langle a_i \rangle \in \mathbb{R}^\omega$ and $\lim a_i = a$, then $[s] \approx {}^*a$.

Proof. For any positive number $\varepsilon \in \mathbb{R}$, $\{i : |a_i - a| < \varepsilon\} \in \mathcal{U}$. Therefore, $[s] - {}^*a$ is infinitesimal. □

Definition

For $f : \mathbb{R} \to \mathbb{R}$, we define ${}^*f : |{}^*\mathfrak{R}| \to |{}^*\mathfrak{R}|$ as follows: for $s \in |\mathfrak{R}^\omega|$,

$$ {}^*f([s]) = [\lambda i. f(s(i))]. $$

The well-definedness of ${}^*f$ follows from

$$ \|s = s'\| \in \mathcal{U} \Rightarrow \|\lambda i. f(s(i)) = \lambda i. f(s'(i))\| \in \mathcal{U}. $$

Also, ${}^*f$ can be obtained from the ultrapower ${}^*\mathfrak{R} \cup \{{}^*f\}$ of

$$ \mathfrak{R} \cup \{f\} = (\mathbb{R}, f, +, \bullet, 0, 1, <). $$

Theorem

$f : \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R}$ $\Leftrightarrow$ for any $x \approx a$, ${}^*f(x) \approx f(a)$.

Proof.

($\Rightarrow$)

- Let $f : \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R}$, and $x = [\langle x_i \rangle] \approx a$. Take any positive number $\varepsilon \in \mathbb{R}$. By the continuity of f, there exists a positive number $\delta \in \mathbb{R}$ such that

$$\forall y \in \mathbb{R}(|y - a| < \delta \to |f(y) - f(a)| < \varepsilon).$$

- Therefore, $\{i : |x_i - a| < \delta\} \subseteq \{i : |f(x_i) - f(a)| < \varepsilon\}$.
- Since $x \approx a$, we have $\{i : |x_i - a| < \delta\} \in \mathcal{U}$.
- Hence, $\{i : |f(x_i) - f(a)| < \varepsilon\} \in \mathcal{U}$. That is, ${}^*f(x) \approx f(a)$.

($\Leftarrow$)

- Suppose that f is not continuous at $a \in \mathbb{R}$.
- That is, there exists a positive number $\varepsilon \in \mathbb{R}$ such that for any $i \in \omega$, there exists x_i such that

$$|x_i - a| < \frac{1}{i+1} \land |f(x_i) - f(a)| \geq \varepsilon$$

- Let $x = [\langle x_i \rangle]$. Then $x \approx a$, $|{}^*f(x) - f(a)| \geq \varepsilon$. In other words, ${}^*f(x) \not\approx f(a)$. □

- Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. By the theorem, for any finite $a \in |{}^*\mathfrak{R}|$,
$$\mathrm{st}({}^*f(a)) = f(\mathrm{st}(a)).$$
- The relationship $S \subseteq \mathbb{R}^n$ of $\mathfrak{R}$ can be naturally extended to the relation ${}^*S$ of ${}^*\mathfrak{R}$. In particular, ${}^*\mathbb{N}$ and ${}^*\mathbb{Q}$ can be viewed as subsets of $|{}^*\mathfrak{R}|$. Moreover, notice that $(\mathfrak{R}, \mathbb{N}, \mathbb{Q})$ is an elementary substructure of $({}^*\mathfrak{R}, {}^*\mathbb{N}, {}^*\mathbb{Q})$.
- Let $N = [\langle 1, 2, 3, \ldots \rangle] \in {}^*\mathbb{N}$. We consider an N-partition of $[0, 1]$ in ${}^*\mathfrak{R}$ as $\{0, 1/N, \ldots, (N-1)/N, N/N\}$.
- Given a standard real number a of $[0, 1]$, take $i \in {}^*\mathbb{N}$ with $i/N \leq a \leq (i+1)/N$, and then we have $a = \mathrm{st}(i/N)$. In other words, any standard real number can be expressed as a non-standard fraction.
- Based on the above observations, many theorems in analysis can be proven by using the non-standard method. Here we will give two examples.

Theorem

A continuous function $f : [0,1] \to \mathbb{R}$ has the maximum value.

Proof. In ${}^*\mathfrak{R}$, consider the following $*$ finite set

$$\{{}^*f(0), {}^*f(1/N), \ldots, {}^*f((N-1)/N), {}^*f(N/N)\}.$$

If its maximum value is ${}^*f(i/N)$, f has the maximum value $\mathrm{st}({}^*f(i/N))$ at $x = \mathrm{st}(i/N)$. □

Remark. Since $(\mathfrak{R}, \mathbb{N})$ is an elementary substructure of $({}^*\mathfrak{R}, {}^*\mathbb{N})$, one can use mathematical induction on ${}^*\mathbb{N}$. For instance, it is provable that any $*$ finite set has the maximal element.

Theorem (Peano)

Let $f : [0,1]^2 \to \mathbb{R}$ be a continuous function. The following differential equation has a solution

$$\mathrm{d}y/\mathrm{d}x = f(x,y), \quad y(0) = 0.$$

Idea of the proof[1] We define $Y : \{0, 1/N, \ldots, N/N\} \to {}^*\mathfrak{R}$ inductively as follows:

$$Y(k/N) = \sum_{i=0}^{k-1} {}^*f(Y(i/N), i/N) \bullet 1/N.$$

Then, $y : [0,1] \to \mathbb{R}$ is defined as follows: given a standard real number a of $[0,1]$, take $k \in {}^*\mathbb{N}$ with $k/N \leq a \leq (k+1)/N$ and set $y(a) = \mathrm{st}(Y(k/N))$. □

Thank you for your attention!

[1] V. Benci and M. Di Nasso, How to Measure the Infinite: Mathematics with Infinite and Infinitesimal Numbers, World Scientific Publishing, 2019.