Logic and Foundation I

Part 2. First-order logic

Kazuyuki Tanaka

BIMSA

November 25, 2023

北京雁栖湖应用数学研究院
Beijing Institute of Mathematical Sciences and Applications

Logic and Foundations I

- Part 1. Equational theory
- Part 2. First order theory
- **Part 3. Model theory**
- Part 4. First order arithmetic and incompleteness theorems

Part 2. Schedule

- Nov. 16, (1) $\forall$-theory and $\forall\exists$-theory
- Nov. 23, (2) Horn theory and reduced products
- Nov. 30, (3) Ultra products and non-standard analysis

Today's topics

1. Recap
2. Homework
3. Horn formula and reduced product
4. Ultraproducts

Recap

- The **theory** of a structure $\mathfrak{A}$, denoted $\mathrm{Th}(\mathfrak{A})$, is the set of sentences true in $\mathfrak{A}$.
- The **elementary diagram** of $\mathfrak{A}$ is $\mathrm{Th}(\mathfrak{A}_\mathrm{A})$.
- $\mathrm{Diag}(\mathfrak{A})$ = the set of atomic sentences and negations of atomic sentences in $\mathrm{Th}(\mathfrak{A}_\mathrm{A})$, is called the **basic diagram**.
- $\mathcal{L}$-structures $\mathfrak{A}$ and $\mathfrak{B}$ are **elementary equivalent**, denoted $\mathfrak{A} \equiv \mathfrak{B}$, if the same $\mathcal{L}$-sentences hold in both structures, that is, $\mathrm{Th}(\mathfrak{A}) = \mathrm{Th}(\mathfrak{B})$
- A structure $\mathfrak{A}$ is an **elementary substructure** of a structure $\mathfrak{B}$, denoted $\mathfrak{A} \prec \mathfrak{B}$, if $\mathfrak{A}$ is a substructure of $\mathfrak{B}$ and the same $\mathcal{L}_A$-sentences hold in both structures, i.e, $\mathrm{Th}(\mathfrak{A}_A) = \mathrm{Th}(\mathfrak{B}_A)$.
- Note that the notion of elementary substructure is stronger than that of elementary equivalence:

$$\mathfrak{A} \cong \mathfrak{B} \Rightarrow \mathfrak{A} \prec \mathfrak{B} \Rightarrow \mathfrak{A} \equiv \mathfrak{B}$$

Recap

Theorem (Tarski-Vaught's criterion)

$\mathfrak{A} \prec \mathfrak{B}$ iff $\mathfrak{A} \subseteq \mathfrak{B}$ and for any formula $\varphi(x, y_1, \ldots, y_m)$ and any $a_1, \ldots, a_m \in A$,

$$\mathfrak{B}_A \models \exists x \varphi(x, a_1, \ldots, a_m) \Rightarrow \text{there exists an } a \in |\mathfrak{A}| \text{ s.t. } \mathfrak{B}_A \models \varphi(a, a_1, \ldots, a_m).$$

Definition

A chain of structures $\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \cdots \subseteq \mathfrak{A}_i \subseteq \cdots (i < \omega)$ is called a **elementary chain** if

$$\mathfrak{A}_0 \prec \mathfrak{A}_1 \prec \cdots \prec \mathfrak{A}_i \prec \cdots \quad (i < \omega)$$

And the structure $\mathfrak{A} = \bigcup_{i<\omega} \mathfrak{A}_i$ is called the **union** of the elementary chain.

Theorem (Elementary chain theorem)

Let $\mathfrak{A}_0 \prec \mathfrak{A}_1 \prec \cdots$ be an elementary chain. Let $\mathfrak{A}$ be the union of the elementary chain. Then for each i, $\mathfrak{A}_i \prec \mathfrak{A}$.

Recap

Definition

For an open formula (a formula without quantifiers) φ,
$\forall x_1 \cdots \forall x_m \varphi$ is called a $\forall$ **formula** (or **universal**, Π_1), and
$\forall x_1 \cdots \forall x_n \exists y_1 \cdots \exists y_m \varphi$ is called a $\forall\exists$ **formula** (or **universal-existential**, Π_2).

A set of $\forall$ sentences is called a $\forall$-**theory** or a **universal theory**,
and a set of $\forall\exists$ sentences is called a $\forall\exists$-**theory** or an **inductive theory**.

Let T be a theory of a language $\mathcal{L}$. We denote the class of all models of T by $\mathrm{Mod(T)}$, i.e.,

$$\mathrm{Mod(T)} = \{\mathfrak{A} : \mathfrak{A} \models \mathrm{T}\}$$

Theorem (Łoś-Tarski)

The following two conditions are equivalent.

1. $\mathrm{Mod(T)}$ *is closed under substructures.*
2. *There exists an* $\forall$*-theory* T' *such that* $\mathrm{Mod(T)} = \mathrm{Mod(T')}$.

Theorem (Chang-Łoś-Suszko)

The followings are equivalent.

(1) $\mathrm{Mod(T)}$ *is closed under the union of chains. That is, if* $\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \cdots$ *is a chain of models of* T*, the union is also a model of* T*.*

(2) *There exists a* $\forall\exists$*-theory* T' *such that* $\mathrm{Mod(T')} = \mathrm{Mod(T)}$*.*

Definition

A theory T is said to be **model complete** if for any model $\mathfrak{A}$, $\mathfrak{B}$ of T,

$$\mathfrak{A} \subseteq \mathfrak{B} \Rightarrow \mathfrak{A} \prec \mathfrak{B}.$$

Lemma

A model-complete theory is a $\forall\exists$*-theory.*

Proof. In a model-complete theory T, a chain of models is an elementary chain, so by the elementary chain theorem, the union is also a model of T. By the Chang-Łoś-Suszko theorem, this theory is a $\forall\exists$ theory.

Problem 6

Let T be a $\forall\exists$ theory, and φ_1, φ_2 be $\forall\exists$ sentences. Now, suppose any model $\mathfrak{A}$ of T can be extended to a model of $T \cup \{\varphi_1\}$ and a model of $T \cup \{\varphi_2\}$. Then show that any model $\mathfrak{A}$ of T can be extended to $T \cup \{\varphi_1, \varphi_2\}$.

Solution:

- Construct a chain $\mathfrak{A} \subseteq \mathfrak{A}_1 \subseteq \mathfrak{A}_2 \subseteq \cdots$ of a $\forall\exists$ theory T such that $\mathfrak{A}_{2i+1}$ is a model of $T \cup \{\varphi_1\}$ and $\mathfrak{A}_{2i+2}$ is a model of $T \cup \{\varphi_2\}$.
- Since $\bigcup_{i \in \mathbb{N}} \mathfrak{A}_i$ is the union of a chain of models $\{\mathfrak{A}_{2i+1}\}$ of a $\forall\exists$ theory $T \cup \{\varphi_1\}$, it is also a model of $T \cup \{\varphi_1\}$, by the Chang-Łoś-Suszko theorem.
- Similarly, since $\bigcup_{i \in \mathbb{N}} \mathfrak{A}_i$ is the union of a chain of models $\{\mathfrak{A}_{2i+2}\}$ of a $\forall\exists$ theory $T \cup \{\varphi_2\}$, it is also a model of $T \cup \{\varphi_2\}$.
- Therefore, $\bigcup_{i \in \mathbb{N}} \mathfrak{A}_i$ is a model of $T \cup \{\varphi_1, \varphi_2\}$. So, any model $\mathfrak{A}$ of T can be extended to $T \cup \{\varphi_1, \varphi_2\}$. □

Problem 6

Let T be a $\forall\exists$ theory, and φ_1, φ_2 be $\forall\exists$ sentences. Now, suppose any model $\mathfrak{A}$ of T can be extended to a model of $T \cup \{\varphi_1\}$ and a model of $T \cup \{\varphi_2\}$. Then show that any model $\mathfrak{A}$ of T can be extended to $T \cup \{\varphi_1, \varphi_2\}$.

Solution:

- Construct a chain $\mathfrak{A} \subseteq \mathfrak{A}_1 \subseteq \mathfrak{A}_2 \subseteq \cdots$ of a $\forall\exists$ theory T such that $\mathfrak{A}_{2i+1}$ is a model of $T \cup \{\varphi_1\}$ and $\mathfrak{A}_{2i+2}$ is a model of $T \cup \{\varphi_2\}$.
- Since $\bigcup_{i \in \mathbb{N}} \mathfrak{A}_i$ is the union of a chain of models $\{\mathfrak{A}_{2i+1}\}$ of a $\forall\exists$ theory $T \cup \{\varphi_1\}$, it is also a model of $T \cup \{\varphi_1\}$, by the Chang-Łoś-Suszko theorem.
- Similarly, since $\bigcup_{i \in \mathbb{N}} \mathfrak{A}_i$ is the union of a chain of models $\{\mathfrak{A}_{2i+2}\}$ of a $\forall\exists$ theory $T \cup \{\varphi_2\}$, it is also a model of $T \cup \{\varphi_2\}$.
- Therefore, $\bigcup_{i \in \mathbb{N}} \mathfrak{A}_i$ is a model of $T \cup \{\varphi_1, \varphi_2\}$. So, any model $\mathfrak{A}$ of T can be extended to $T \cup \{\varphi_1, \varphi_2\}$. □

Problem 7 Homework

In a model-complete theory, show that for every formula, there exists an equivalent $\forall$ formula. (Hint. See the proof of (1)⇒(2) in the Łoś-Tarski theorem.)

Solution:

- Let φ be formula. We may assume that φ is a sentence by replacing the free variables contained in φ with new constants. Then, if a $\forall$ sentence equivalent to that sentence is found, by replacing the new constants in it with the original variables, we will obtain a $\forall$ formula equivalent to the original formula.
- Furthermore, we may assume that $T \cup \{\varphi\}$ is consistent. Otherwise, any sentence $\perp$ expressing a contradiction is equivalent to φ on T.
- Let $T' = \{\sigma : \sigma$ is a $\forall$ statement, and $T \cup \{\varphi\} \vdash \sigma\}$.
- Now, let $\mathfrak{A}$ be an arbitrary model of $T \cup T'$. In the same way as the proof of the Łos-Tarski theorem, we can construct a model $\mathfrak{B}_A$ of $\mathrm{Diag}(\mathfrak{A}) \cup \mathrm{T} \cup \{\varphi\}$. Then by the model completeness of T, we have $\mathfrak{A} \prec \mathfrak{B}_A$. Therefore, $\mathfrak{A} \models \varphi$.
- By the completeness theorem, $T \cup T' \vdash \varphi$. Therefore, there exists a finite subset $\{\sigma_1, \sigma_2, \ldots, \sigma_n\} \subset T'$ such that $T \vdash (\sigma_1 \wedge \cdots \wedge \sigma_n) \to \varphi$. Since $(\sigma_1 \wedge \cdots \wedge \sigma_n)$ can be easily transformed into an equivalent $\forall$ sentence σ, we have $T \vdash \sigma \leftrightarrow \varphi$.

Problem 7 Homework

In a model-complete theory, show that for every formula, there exists an equivalent $\forall$ formula. (Hint. See the proof of (1)⇒(2) in the Łoś-Tarski theorem.)

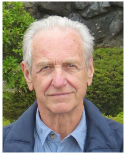
Solution:

- Let φ be formula. We may assume that φ is a sentence by replacing the free variables contained in φ with new constants. Then, if a $\forall$ sentence equivalent to that sentence is found, by replacing the new constants in it with the original variables, we will obtain a $\forall$ formula equivalent to the original formula.
- Furthermore, we may assume that $T \cup \{\varphi\}$ is consistent. Otherwise, any sentence $\perp$ expressing a contradiction is equivalent to φ on T.
- Let $T' = \{\sigma : \sigma$ is a $\forall$ statement, and $T \cup \{\varphi\} \vdash \sigma\}$.
- Now, let $\mathfrak{A}$ be an arbitrary model of $T \cup T'$. In the same way as the proof of the Łos-Tarski theorem, we can construct a model $\mathfrak{B}_A$ of $\mathrm{Diag}(\mathfrak{A}) \cup \mathrm{T} \cup \{\varphi\}$. Then by the model completeness of T, we have $\mathfrak{A} \prec \mathfrak{B}_A$. Therefore, $\mathfrak{A} \models \varphi$.
- By the completeness theorem, $T \cup T' \vdash \varphi$. Therefore, there exists a finite subset $\{\sigma_1, \sigma_2, \ldots, \sigma_n\} \subset T'$ such that $T \vdash (\sigma_1 \wedge \cdots \wedge \sigma_n) \to \varphi$. Since $(\sigma_1 \wedge \cdots \wedge \sigma_n)$ can be easily transformed into an equivalent $\forall$ sentence σ, we have $T \vdash \sigma \leftrightarrow \varphi$.

Horn formula

- The **Horn formula** was introduced by A. Horn in the early 1950s for mathematical interest (related with direct product) as we will introduce today.
- However, in the 1970s, R. Kowalski discovered an efficient procedure for proving Horn formulas. Based on this idea, the theory and applications of logic programming rapidly developed.[a]
- Horn formulas have become widely known as the logic of artificial intelligence.

A. Horn



R. Kowalski

[a]Robert Kowalski: A Short Story of My Life and Work https://www.doc.ic.ac.uk/ rak/history.pdf

We fix a language $\mathcal{L}$.

Definition

- For atomic formulas $\theta_i\ (i < n)$, $\theta_0 \vee \neg\theta_1 \vee \cdots \vee \neg\theta_n$ or $\neg\theta_1 \vee \cdots \vee \neg\theta_n$ is called a **basic Horn formula**.
- A formula constructed from the basic Horn formulas by using only $\wedge$, $\forall$, and $\exists$ is called a **Horn formula**.
- The set of Horn sentences is called a **Horn theory**.

A basic Horn formula can be expressed as follows, which is easier to use in applications:

$$\theta_1 \wedge \cdots \wedge \theta_n \to \theta_0$$

or

$$\theta_1 \wedge \cdots \wedge \theta_n \to \bot,$$

where $\bot$ denotes a contradiction.

Example

- The theory of regular rings, which adds the axiom $\forall x \forall y (xyx = x)$ to ring theory, is a Horn theory.
- The theory of integral domain (commutative ring theory + $\forall x \forall y (x \bullet y = 0 \to x = 0 \lor y = 0)$) and field theory (commutative ring theory + $\forall x \exists y (x \neq 0 \to xy = 1)$) is not Horn theory.

The models of a Horn theory is closed under "reduced products", which is a generalization of direct product.

Before introducing reduced products, we begins with some preliminary definitions.

Definition

Let I be a non-empty set. $\mathcal{F} \subseteq \mathcal{P}(I)$ is said to be **filter** on I if the following are satisfied.

(1) $\varnothing \notin \mathcal{F}$, $I \in \mathcal{F}$.

(2) $X \in \mathcal{F}, X \subseteq Y \subseteq I \Rightarrow Y \in \mathcal{F}$.

(3) $X, Y \in \mathcal{F} \rightarrow X \cap Y \in \mathcal{F}$.

Problem 8

Let I be an infinite set. Show the following.

1. The collection of all finite subsets of I is not a filter.
2. The collection of all infinite subset of I is not a filter.
3. The collection of subsets of I whose complement is finite is a filter, which is called a **Fréchet filter**.
4. For each $i \in I$, the collection of all subsets of I containing i, $\{X \subseteq I : i \in X\}$ is a filter, which is called a **principal filter**.

Lemma

If $S \subset \mathcal{P}(I)$ has the **finite intersection property**: *for any finite subset $\{J_1, \dots J_n\} \subset S$,*

$$J_1 \cap \cdots \cap J_n \neq \varnothing,$$

then there exists a filter $\mathcal{F}$ including S.

Proof. Let $\mathcal{F} = \{X \subseteq I : J_1 \cap \cdots \cap J_n \subset X \text{ for some } \{J_1, \dots, J_n\} \subset S\}$. □

Definition

Let $\mathfrak{A}_i = (A_i, \mathtt{f}^{\mathfrak{A}_i}, \dots, \mathrm{R}^{\mathfrak{A}_i}, \dots)\ (i \in I)$ be an $\mathcal{L}$-structure.
Let $\mathcal{F}$ be a filter on I. Then, we define the binary relation $\approx_{\mathcal{F}}$ on $\prod A_i$ as follows

$$a \approx_{\mathcal{F}} b \quad \Leftrightarrow \quad \{i \in I : a(i) = b(i)\} \in \mathcal{F}.$$

Lemma

$\approx_{\mathcal{F}}$ is an equivalence relation.

Proof.

- The laws of reflection and symmetry are clear from the definitions.
- To show the transitive law, we assume $a \approx_{\mathcal{F}} b$, $b \approx_{\mathcal{F}} c$.
- By definition,

$$\{i \in I : a(i) = b(i)\} \in \mathcal{F} \text{ and } \{i \in I : b(i) = c(i)\} \in \mathcal{F}.$$

- On the other hand,

$$\{i \in I : a(i) = c(i)\} \supseteq \{i \in I : a(i) = b(i)\} \cap \{i \in I : b(i) = c(i)\}$$

- By conditions (2) $X \in \mathcal{F}, X \subseteq Y \subseteq I \Rightarrow Y \in \mathcal{F}$ and (3) $X, Y \in \mathcal{F} \to X \cap Y \in \mathcal{F}$ of the definition of filter, we have

$$\{i \in I : a(i) = c(i)\} \in \mathcal{F}.$$

Therefore, $a \approx_{\mathcal{F}} c$. □

For $a_1, \dots, a_n \in \prod A_i$, we set $\|\varphi(a_1, \dots, a_n)\| := \{i \in I : \mathfrak{A}_i \models \varphi(a_1(i), \dots, a_n(i))\}$.
$\mathfrak{A}_i$ here is strictly $\mathfrak{A}_{iA_i}$. But for simplicity, we write $\mathfrak{A}$ for $\mathfrak{A}_A$ if it is clear from the context.

Lemma

If $a_1 \approx_{\mathcal{F}} b_1, \dots, a_n \approx_{\mathcal{F}} b_n$, *we have*

$$\|\mathtt{f}(a_1, \dots, a_n) = \mathtt{f}(b_1, \dots, b_n)\| \in \mathcal{F},$$

$$\|\mathrm{R}(a_1, \dots, a_n)\| \in \mathcal{F} \Leftrightarrow \|\mathrm{R}(b_1, \dots, b_n)\| \in \mathcal{F}.$$

Proof.
This can be derived from the following and the definition of filter.

$$\bigcap_{k \leq n} \{i \in I : a_k(i) = b_k(i)\} \subseteq \|\mathtt{f}(a_1, \dots, a_n) = \mathtt{f}(b_1, \dots, b_n)\|,$$

$$\bigcap_{k \leq n} \{i \in I : a_k(i) = b_k(i)\} \cap \|\mathrm{R}(a_1, \dots, a_n)\| \subseteq \ |\mathrm{R}(b_1, \dots, b_n)\|.$$

□

Therefore, $\approx_{\mathcal{F}}$ is a congruence relation on $\prod A_i$.

- We can define the quotient structure in the same way as for the algebraic structure.
- That is, its domain is the set of equivalence classes denoted $\prod A_i/{\approx_{\mathcal{F}}}$ or $\prod A_i/\mathcal{F}$, and the value of a function f and the truth value of a relation R is uniquely determined on the equivalence classes regardless of choice of representative elements.

Definition

Let $\mathfrak{A}_i = (A_i, \mathtt{f}^{\mathfrak{A}_i}, \ldots, \mathrm{R}^{\mathfrak{A}_i}, \ldots)\,(i \in I)$ be $\mathcal{L}$-structures. Let $\mathcal{F}$ be a filter on I. Then, the following $\mathcal{L}$-structure is called the **reduced product** of $\mathfrak{A}_i$, denoted by $\prod \mathfrak{A}_i/\mathcal{F}$.

$$\left(\prod A_i/\mathcal{F}, \mathtt{f}^{\prod \mathfrak{A}_i/\mathcal{F}}, \ldots, \mathrm{R}^{\prod \mathfrak{A}_i/\mathcal{F}}, \ldots\right)$$

- For a non-empty set I, $\mathcal{F} = \{I\}$ is a filter and $\prod \mathfrak{A}_i/\mathcal{F} \cong \prod \mathfrak{A}_i$.
  In other words, the direct product is also one kind of the reduced products.
- For the principal filter $\mathcal{F} = \{X \subseteq I : k \in X\}$, $\prod \mathfrak{A}_i/\mathcal{F} \cong \mathfrak{A}_k$.

Lemma

If φ is a formula obtained from the atomic formula with $\wedge$ and $\exists$, then for $a_1, \ldots, a_n \in \prod A_i$,

$$\prod \mathfrak{A}_i/\mathcal{F} \models \varphi([a_1], \ldots, [a_n]) \Leftrightarrow \|\varphi(a_1, \ldots, a_n)\| \in \mathcal{F}.$$

Proof. By induction on the construction of formulas.

- If φ is an atomic formula, it is clear by the definition.
- If $\varphi = \psi_1 \wedge \psi_2$, it follows from the induction hypo. and the closedness of filter under $\cap$.
- Let $\varphi = \exists x \psi(x)$. For simplicity, we do not display parameters $a_1, \ldots, a_n$ in φ.

$$\begin{aligned}\prod \mathfrak{A}_i/\mathcal{F} \models \exists x \psi(x) &\Leftrightarrow \text{ for some } a \in \prod \mathfrak{A}_i \text{ , } \prod \mathfrak{A}_i/\mathcal{F} \models \psi([a]) \\ &\Leftrightarrow \text{ for some } a \in \prod \mathfrak{A}_i \text{ , } \|\psi(a)\| \in \mathcal{F} \quad \text{(induction hypothesis)} \\ &\Rightarrow \|\exists x \psi(x)\| \in \mathcal{F} \quad (\because \|\psi(a)\| \subseteq \|\exists x \psi(x)\|).\end{aligned}$$

- Conversely, let $\|\exists x \psi(x)\| \in \mathcal{F}$. By the axiom of choice, we take $a \in \prod A_i$ such that for each $i \in \|\exists x \psi(x)\|$, $\mathfrak{A}_i \models \psi(a(i))$. Then, $\|\psi(a)\| \in \mathcal{F}$. By the induction hypothesis, $\prod \mathfrak{A}_i/\mathcal{F} \models \psi([a])$. Therefore, $\prod \mathfrak{A}_i/\mathcal{F} \models \exists x \psi(x)$. □

Lemma

Let $\varphi(x_1, \ldots, x_n)$ *be a basic Horn formula, then for* $a_1, \ldots, a_n \in \prod A_i$*, we have*

$$\|\varphi(a_1, \ldots, a_n)\| \in \mathcal{F} \Rightarrow \prod \mathfrak{A}_i/\mathcal{F} \models \varphi([a_1], \ldots, [a_n]).$$

Proof.

- For simplicity, we do not display parameters $a_1, \ldots, a_n \in \prod A_i$ in the formula.
- Let φ be a basic horn sentence $(\theta_0 \vee)\neg\theta_1 \vee \cdots \vee \neg\theta_n$, where θ_i $(i < n)$ are atomic sentences. We show a contradiction by assuming $\underline{①\|\varphi\| \in \mathcal{F}}$ and $\underline{②\prod \mathfrak{A}_i/\mathcal{F} \not\models \varphi}$.
- By ②, since $\prod \mathfrak{A}_i/\mathcal{F} \models \theta_1 \wedge \cdots \wedge \theta_n$, by the last lemma, we have $\|\theta_1 \wedge \cdots \wedge \theta_n\| \in \mathcal{F}$.
  - If φ does not contain θ_0, we have $\varnothing = \|\varphi\| \cap \|\theta_1 \wedge \cdots \wedge \theta_n\| \in \mathcal{F}$, which violates the condition of a filter .
  - If φ contains θ_0, we have $\|\theta_0\| = \|\varphi\| \cap \|\theta_1 \wedge \cdots \wedge \theta_n\| \in \mathcal{F}$. Thus by the last lemma, we have $\prod \mathfrak{A}_i/\mathcal{F} \models \theta_0$, which conflicts with the assumption $\prod \mathfrak{A}_i/\mathcal{F} \not\models \varphi$ □

Lemma

Let $\varphi(x_1, \ldots, x_n)$ be a Horn formula, then for $a_1, \ldots, a_n \in \prod A_i$,

$$\|\varphi(a_1, \ldots, a_n)\| \in \mathcal{F} \Rightarrow \prod \mathfrak{A}_i/\mathcal{F} \models \varphi([a_1], \ldots, [a_n]).$$

Proof. By induction on the construction of a Horn formula with $\wedge$, $\forall$, and $\exists$.
For the basic Horn formula, it follows from the last lemma. Formulas $\varphi \wedge \psi$ and $\exists x \varphi(x)$ are treated in the lemma in Page 18.
For a formula $\forall x \varphi(x)$,

$$\begin{aligned} \|\forall x \varphi(x)\| \in \mathcal{F} &\Rightarrow \text{for all } a \in \prod A_i,\ \|\varphi(a)\| \in \mathcal{F} \\ &\Rightarrow \text{for all } a \in \prod A_i,\ \prod \mathfrak{A}_i/\mathcal{F} \models \varphi([a]) \\ &\Leftrightarrow \prod \mathfrak{A}_i/\mathcal{F} \models \forall x \varphi(x) \quad \square \end{aligned}$$

- The above lemma shows that the Horn formula preserves reduced products, that is, a reduced product of models of a Horn formula becomes a model of the original Horn formula again.
- Therefore, the class of models of a Horn theory is closed under reduced products, especially under direct products. Then the converse is also true in the following sense.

Theorem (Keisler-Galvin)

The following are equivalent:

(1) $\mathrm{Mod}(\mathrm{T})$ *is closed under reduced products.*

(2) *There exists a Horn theory* T' *such that* $\mathrm{Mod}(\mathrm{T}) = \mathrm{Mod}(\mathrm{T}')$.

A proof can be found in Chang-Keisler's classic textbook *Model Theory*.

(Example) The product of regular rings is a regular ring.

(Exercise) Show that the class of Boolean algebras with atoms (non-atomless) is closed under direct products but not under reduced products.

- The theory of Boolean algebra is a ∀-theory. "Boolean algebra has an atom a" is expressed by the following ∃∀-sentence.

$$\exists a \forall x (a \neq 0 \land (a \bullet x = x \to x = a \lor x = 0)).$$

- In a direct product $\prod \mathfrak{A}_i$ of such Boolean algebras, consider a function f whose value is an atom $a \in |\mathfrak{A}_i|$ for only one i and 0 elsewhere. Then, f becomes an atom of $\prod \mathfrak{A}_i$.
- On the other hand, consider the reduced product $\prod \mathfrak{A}_i/\mathcal{F}$ with the Fréchet filter $\mathcal{F}$. Assume that it has an atom $[g]$. Since it is not zero $0^{\prod \mathfrak{A}_i/\mathcal{F}}$, it takes a value other than 0 on an infinite set $J \subseteq I$. Now divide J into two infinite sets J_1 and J_2. Let h be the function obtained from g by replacing its values on J_2 with 0. Then we have

$$[g] \bullet [h] = [h], \ [h] \neq [g], \ [h] \neq 0,$$

which contradicts the assumption that $[g]$ is an atom.
- It is not easy to describe which sentences preserve the direct product. In fact, Machover (1960) showed it is not computable.

(Exercise) Show that the class of Boolean algebras with atoms (non-atomless) is closed under direct products but not under reduced products.

- The theory of Boolean algebra is a $\forall$-theory. "Boolean algebra has an atom a" is expressed by the following $\exists\forall$-sentence.

$$\exists a \forall x (a \neq 0 \land (a \bullet x = x \to x = a \lor x = 0)).$$

- In a direct product $\prod \mathfrak{A}_i$ of such Boolean algebras, consider a function f whose value is an atom $a \in |\mathfrak{A}_i|$ for only one i and 0 elsewhere. Then, f becomes an atom of $\prod \mathfrak{A}_i$.
- On the other hand, consider the reduced product $\prod \mathfrak{A}_i/\mathcal{F}$ with the Fréchet filter $\mathcal{F}$. Assume that it has an atom $[g]$. Since it is not zero $0^{\prod \mathfrak{A}_i/\mathcal{F}}$, it takes a value other than 0 on an infinite set $J \subseteq I$. Now divide J into two infinite sets J_1 and J_2. Let h be the function obtained from g by replacing its values on J_2 with 0. Then we have

$$[g] \bullet [h] = [h], \ [h] \neq [g], \ [h] \neq 0,$$

which contradicts the assumption that $[g]$ is an atom.
- It is not easy to describe which sentences preserve the direct product. In fact, Machover (1960) showed it is not computable.

- A sentence with several $\forall$ in front of a basic Horn formula is called a **$\forall$-Horn sentence** (or simply called a Horn sentence in some literature). A collection of such sentences is called a **$\forall$-Horn theory** (or simply a Horn theory).
- A $\forall$-Horn theory is a nice extension of equational theory. The following theorem is a counter part of Birkhoff's equational class theorem. It can be proven similarly, and we leave the details to the reader.

Theorem

Let $\mathcal{K}$ be a class of $\mathcal{L}$-structures, then the following are equivalent:

(1) *$\mathcal{K}$ is closed under direct products, substructures, and isomorphic images.*

(2) *$\mathcal{K}$ is closed under reduced products, substructures, and isomorphic images.*

(3) *There exists a $\forall$-Horn theory T such that $\mathrm{Mod(T)} = \mathcal{K}$.*

Ultraproducts

In the following, we will consider the necessary and sufficient conditions for a class of structures to be axiomatized by first order logic, that is, be expressed as $\mathrm{Mod(T)}$.

Definition

A class $\mathcal{K}$ of $\mathcal{L}$-structures is called an **elementary class** if there exists a set T of sentences such that $\mathcal{K} = \mathrm{Mod(T)}$. In this case, we write

$$\mathcal{K} \in \mathrm{EC}_{\Delta}.$$

To characterize elementary classes, we use a kind of reduced product called "ultraproduct". To define it, we first introduce an ultrafilter.

Definition

The filter $\mathcal{F}$ on I is an **ultrafilter** (maximal filter) if the following properties are satisfied.

$$\forall X \subset I (X \in \mathcal{F} \vee I - X \in \mathcal{F}).$$

Lemma

Every filter $\mathcal{F}$ can be expanded to an ultrafilter $\mathcal{U}$.

Proof. Consider the class of all filters including a given filter $\mathcal{F}$.
Since it is closed under the union of chains, by Zorn's lemma, there is a maximal filter $\mathcal{U}$ which is an ultrafilter. □

A principal filter is an ultrafilter. There exists an ultrafilter which is **non-principal**.

Lemma

There exists a non-principal ultrafilter $\mathcal{U}$ on any infinite set I.

Proof.
Let I be an infinite set, and $\mathcal{F}$ be a Fréchet filter on it (a subset of I whose complement is finite). By the above lemma, an ultrafilter $\mathcal{U}$ can be obtained by expanding $\mathcal{F}$. Then $\mathcal{U}$ is non-principal, since for each $i \in I$, $I - \{i\} \in \mathcal{F} \subseteq \mathcal{U}$, so we have $\{i\} \notin \mathcal{U}$.

Stone's representation theorem

We prove Stone's representation theorem using an ultrafilter.

Theorem (Stone's representation theorem)

For any Boolean algebra $\mathfrak{B}$, there exists a set X, and $\mathfrak{B}$ can be embedded in the power set algebra $\mathfrak{P}(X)$.
In particular, if $\mathfrak{B}$ is finite, it is isomorphic to $\mathfrak{P}(X)$.

Proof.

- Let $\mathfrak{B} = (B, \vee, \wedge, \neg, 0, 1)$ be a Boolean algebra. Filters, Ultrafilters, and others can naturally be defined for a subset $F \subseteq B$ with the ordering $x \leq y \Leftrightarrow x \wedge y = x$. Let X be the set of all ultrafilters of B and $\mathcal{P}(X)$ be its power set.
- Now, $f : B \to \mathcal{P}(X)$ is defined as follows:
  For each $b \in B$, $f(b)$ is the set of all ultrafilters containing b. We show f is injective.
  If $a \neq b$, then $\underline{①a \wedge (\neg b) \neq 0}$ or $\underline{②(\neg a) \wedge b \neq 0}$.

  Case ①. Since $\{a, \neg b\}$ has the finite intersection property, it can be extended to an ultrafilter $U \subseteq B$. Thus, $U \in f(a)$ and $U \notin f(b)$, and we have $f(a) \neq f(b)$.
  Case ② can be treated similarly.

- Furthermore, by the property of filter F: $a \wedge b \in F \Leftrightarrow a \in F$ and $b \in F$, we have
$$f(a \wedge b) = f(a) \cap f(b).$$
- Also, by the property of the ultrafilter U: $a \notin U \Leftrightarrow \neg a \in U$, we have
$$f(\neg a) = X - f(a).$$
Thus, $f : B \to \mathcal{P}(X)$ is embedding.
- If $\mathfrak{B}$ is finite, any ultrafilter must be a principal filter. And its generator is an atom (non-zero minimal element) in $\mathfrak{B}$. So, let X be the set of atoms. It is easy to see that $\mathfrak{B}$ and $\mathfrak{P}(X)$ are isomorphic. □

Ultraproducts

Definition (Ultraproduct)

The reduced product $\prod \mathfrak{A}_i/\mathcal{U}$ for an ultrafilter $\mathcal{U}$ is called an **ultraproduct**.

Theorem (Łos)

Let $\mathcal{U}$ be an ultrafilter. For any formula $\varphi(x_1, \dots, x_n)$ and $a_1, \dots, a_n \in \prod A_i$,
$\prod \mathfrak{A}_i/\mathcal{U} \models \varphi([a_1], \dots, [a_n]) \Leftrightarrow \|\varphi(a_1, \dots, a_n)\| \in \mathcal{U}$.

Proof. By induction on the construction of formulas. The atomic formulas and formulas beginning with $\wedge$ and $\exists$ are treated in Page 18.
Then we only need to treat the case of negation, $\neg\varphi$ since $\vee$ and $\forall$ can be expressed by $\wedge, \exists$ and negation $\neg$.

$$\begin{aligned} \prod \mathfrak{A}_i/\mathcal{U} \models \neg\varphi &\Leftrightarrow \prod \mathfrak{A}_i/\mathcal{U} \not\models \varphi \\ &\Leftrightarrow \|\varphi\| \notin \mathcal{U} \quad (\because \text{induction hypothesis}) \\ &\Leftrightarrow \|\neg\varphi\| \in \mathcal{U} \quad (\because \text{maximality of } \mathcal{U}). \quad \square \end{aligned}$$

By applying the above theorem, we obtain another proof of compactness theorem.

Corollary (Compactness theorem)

A theory T has a model iff any finite subset of T has a model.

- The necessity is clear and we show the sufficiency.
- Let I be the set of finite subsets of T. For each $\varphi \in T$, let $J_\varphi = \{i \in I : \varphi \in i\}$. Then $\{J_\varphi : \varphi \in T\}$ has the finite intersection property since $\{\varphi_1, \ldots, \varphi_n\} \in J_{\varphi_1} \cap \cdots \cap J_{\varphi_n}$.
- There exists an ultrafilter $\mathcal{U} \supseteq \{J_\varphi : \varphi \in T\}$ by the lemma on Page 14 and the first lemma on Page 25.
- Let $\mathfrak{A}_i$ be a model for each $i \in I$ and $\mathfrak{A} = \prod \mathfrak{A}_i/\mathcal{U}$. We show that $\mathfrak{A}$ is a model of T.
- First, take an arbitrary $\varphi \in T$. Since

$$i \in J_\varphi \Rightarrow \varphi \in i \Rightarrow \mathfrak{A}_i \models \varphi,$$

  we have $J_\varphi \subseteq \{i : \mathfrak{A}_i \models \varphi\}$. Since $J_\varphi \in \mathcal{U}$, $\|\varphi\| = \{i : \mathfrak{A}_i \models \varphi\} \in \mathcal{U}$.
- By the Łos Theorem, we have $\mathfrak{A} \models \varphi$.

Problem 9: Homework

Use an ultraproduct to show that any field $\mathcal{F}$ has algebraic closure $\overline{\mathcal{F}}$.

(Hint. Let $\mathcal{F}_P$ be the splitting field of a polynomial P, and for each $Q \in \mathcal{F}[X]$,

$$J_Q = \{P \in \mathcal{F}[X] : Q \text{ is splitted into a product of linear expressions over } \mathcal{F}_P\}.$$

Then, let $\mathcal{U}$ be an ultrafilter containing $\{J_Q : Q \in \mathcal{F}[X]\}$, and consider the ultraproduct $\prod \mathcal{F}_P/\mathcal{U}$.)

Thank you for your attention!