Logic and Foundation I

Part 2. First-order logic

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Logic and Foundations I

- **Part 1. Equational theory**
- **Part 2. First order theory**
- **Part 3. Model theory**
- **Part 4. First order arithmetic and incompleteness theorems**

Part 2. Schedule

- Oct. 26, (1) First order logic: formal system GT and structures
- Nov. 2, (2) Gödel's completeness theorem and applications
- Nov. 9, (3) Miscellaneous

Today's topics

1. Recap
2. Conservative extension
3. Second order logic
   - General structures

Recap

- First-order logic is developed in the common logical symbols and specific mathematical symbols. Major logical symbols are propositional connectives, quantifiers $\forall x$ and $\exists x$ and equality $=$. The set of mathematical symbols to use is called a **language**.
- A **structure** in language $\mathcal{L}$ (simply, a $\mathcal{L}$-structure) is defined as a non-empty set A equipped with an interpretation of the symbols in $\mathcal{L}$.
- A **term** is a symbol string to denote an element of a structure. A **formula** is a symbol string to describe a property of a structure. A formula without free variables is called a **sentence**.
- "A sentence φ is **true** in $\mathcal{A}$, written as $\mathcal{A} \models \varphi$" is defined by Tarski's clauses. The truth of a formula with free variables is defined by the truth of its universal closure.
- A set of sentences in the language $\mathcal{L}$ is called a **theory**. $\mathcal{A}$ is a **model** of T, denoted by $\mathcal{A} \models T$, if $\forall \varphi \in T\ (\mathcal{A} \models \varphi)$.
- We say that φ holds in T, written as $T \models \varphi$, if $\forall \mathcal{A}(\mathcal{A} \models T \to \mathcal{A} \models \varphi)$.

Definition (Gentzen-Tait system $\mathrm{GT}(T)$ of a theory T)

A sequent $\varphi_1, \ldots, \varphi_n$ (i.e., a multi-set of formulas) intuitively means $\varphi_1 \vee \cdots \vee \varphi_n$.
A formula φ is automatically transformed into the negation normal form, i.e., constructed from atomic formulas or their negations by means of $\wedge$, $\vee$, $\forall$, and $\exists$.

Axioms

(0) φ (where $\varphi \in T$)

(1) Law of excluded middle: $\neg\psi, \psi$ (where ψ is an atomic formula)

(2) axioms of equations: (i) $x = x$, (ii) $x \neq y, y = x$, etc.

Inference rules

$$\frac{\Gamma, \varphi, \psi}{\Gamma, \varphi \vee \psi}\ (\vee), \quad \frac{\Gamma, \varphi \quad \Gamma, \psi}{\Gamma, \varphi \wedge \psi}\ (\wedge), \quad \frac{\Gamma, \varphi(t)}{\Gamma, \exists x \varphi(x)}\ (\exists), \quad \frac{\Gamma, \varphi(x)}{\Gamma, \forall x \varphi(x)}\ (\forall)(x \text{ is not free in } \Gamma)$$

$$\frac{\Gamma}{\Delta}\ (\text{weak})(\Gamma \text{ is a subsequence of } \Delta), \quad \frac{\Gamma, \neg A \quad \Gamma, A}{\Gamma}\ (\text{cut})$$

Definition

- A **proof tree** in the system $\mathsf{GT}(T)$ is a finite tree in which each vertex is labelled with a sequent so that a sequent at each top vertex (leaf) is an axiom, and the sequents of adjacent nodes express an inference rule. See an example below.
- If there is a proof tree rooted at a sequent Γ, we write it as $T \vdash \Gamma$. Such a tree is called a **proof** of $T \vdash \Gamma$ (or a **proof** of Γ in T).
- If $T = \varnothing$ or T is clear from the context, we omit T and write $\vdash \Gamma$.

Example 5

For any term t,

$$\dfrac{\dfrac{\dfrac{x = x}{\forall x(x = x)}\,(\forall)}{\forall x(x = x), t = t}\,(\text{weak}) \qquad \dfrac{t \neq t, t = t}{\exists x(x \neq x), t = t}\,(\exists)}{t = t}\,(\text{cut})$$

Lemma

$\vdash \neg\varphi, \varphi$ *for any formula* φ.

Definition

T is said to be **inconsistent** if $T \vdash$ (i.e., T proves the empty sequent). Otherwise, T is said to be **consistent**.

Lemma

For any sentence φ, $T \cup \{\neg\varphi\}$ is inconsistent $\Leftrightarrow T \vdash \varphi$.

Lemma (complete Henkin extension)

Let T be a consistent theory in a language $\mathcal{L}$. Then, there are a set C of new constants (called a **Henkin constants***) and a theory S in $\mathcal{L}' = \mathcal{L} \cup C$ such that:*

(0) *$T \subseteq S$ and S is also consistent.*

(1) *For each $\mathcal{L}'$-sentence $\exists x\varphi(x)$, there exists a $\mathsf{c} \in C$ such that $\neg\exists x\varphi(x) \vee \varphi(\mathsf{c})$ (called a* **Henkin axiom***) belongs to S. In other words, if $S \vdash \exists x\varphi(x)$, there exists a c such that $S \vdash \varphi(\mathsf{c})$.*

(2) *For any sentence φ in $\mathcal{L}'$, $\varphi \in S$ or $\neg\varphi \in S$.*

By Zorn's lemma, S exists as a maximal consistent set $\supset T \cup H$ (the Henkin axioms).

Theorem

Any consistent theory has a model.

Proof.

- For a consistent theory T, we will construct a model of T using the set of Henkin constants C and the complete Henkin extension S.
- First, define the congruence relation $\mathtt{c} \approx \mathtt{d}$ on C by $(\mathtt{c} = \mathtt{d}) \in S$.
- We define $\mathcal{L}$-structure $\mathfrak{A} = (A, \mathtt{f}^{\mathfrak{A}}, \dots, \mathrm{R}^{\mathfrak{A}}, \dots)$ as follows:

$$A := C/\approx \ : \text{the set of equivalence classes } \{[\mathtt{c}] : \mathtt{c} \in C\},$$

$$\mathtt{f}^{\mathfrak{A}}([\mathtt{c}_0], [\mathtt{c}_1], \dots, [\mathtt{c}_{m-1}]) = [\mathtt{d}] \quad \Leftrightarrow \quad (\mathtt{f}(\mathtt{c}_0, \mathtt{c}_1, \dots, \mathtt{c}_{m-1}) = \mathtt{d}) \in S,$$
$$\mathrm{R}^{\mathfrak{A}}([\mathtt{c}_0], [\mathtt{c}_1], \dots, [\mathtt{c}_{n-1}]) \quad \Leftrightarrow \quad \mathrm{R}(\mathtt{c}_0, \mathtt{c}_1, \dots, \mathtt{c}_{n-1}) \in S.$$

- Then, for any formula $\varphi(x_0, x_1, \dots, x_{n-1})$ in $\mathcal{L}$,

$$\varphi([\mathtt{c}_0], [\mathtt{c}_1], \dots, [\mathtt{c}_{n-1}]) \in \mathrm{Th}(\mathfrak{A}_A) \quad \Leftrightarrow \quad \varphi(\mathtt{c}_0, \mathtt{c}_1, \dots, \mathtt{c}_{n-1}) \in S.$$

- Therefore, $\mathfrak{A}$ is a model of S, and it is also a model of T. □

Theorem (Gödel-Henkin's completeness theorem)

$T \vdash \varphi \Leftrightarrow T \models \varphi$.

Proof.

($\Rightarrow$). Suppose there exists a proof tree P of $T \vdash \varphi$.

- We can easily show that all sequents that appear in P are true in any model $\mathfrak{A}$ of T.

($\Leftarrow$). Assume $T \nvdash \varphi$. We also assume that φ is a sentence.

- Then, $T \cup \{\neg\varphi\}$ is consistent, and so $T \cup \{\neg\varphi\}$ has a model, i.e., $T \not\models \varphi$. □

Theorem (Compactness theorem)

A theory T has a model if and only if any finite subset of T has a model.

Proof. $\Rightarrow$ is obvious. So we only show $\Leftarrow$.

- By way of contradiction, suppose T has no model. Then, $T \models$ (the empty sequent). By the completeness theorem, we also have $T \vdash$. Since a proof tree includes only finitely many axioms, there is a finite set $T' \subset T$ such that $T' \vdash$. Therefore, by the completeness theorem, $T' \models$, that is, some finite subset of T has no model. □

Theorem (Löwenheim-Skolem's downward theorem)

A consistent theory in the language $\mathcal{L}$ has a model whose cardinality is less than or equal to the cardinality of $\mathcal{L}$ or the countable infinity.

Proof. By the Completeness Theorem and lemmas, the cardinality of the Henkin constants is no larger than the cardinality of $\mathcal{L}$ and the countable infinity. Since $\mathfrak{A}$ is constructed from the equivalence classes, the cardinality of $|\mathfrak{A}|$ is less than or equal to them. □

Theorem (Löwenheim-Skolem-Tarski's upward theorem)

If a theory T_0 in a language $\mathcal{L}$ has an infinite model, then it has a model with an arbitrary cardinality κ greater than or equal to the cardinality of $\mathcal{L}$.

Proof. Let $\mathfrak{A}$ be an infinite model of the theory T_0 in $\mathcal{L}$ and κ a cardinal number greater than or equal to the cardinality of $\mathcal{L}$. Let C be a set of new constants with size κ. Let $T = T_0 \cup \{\mathtt{c} \neq \mathtt{d} : \mathtt{c}$ and $\mathtt{d}$ are two distinct constants belonging to $C\}$. Then, any finite subset T' of T has a model $\mathfrak{A}$ with an appropriate interpretation of constants in C so that a finite number of $\mathtt{c} \neq \mathtt{d}$ contained in T' hold. Therefore, by the compactness theorem, T has a model. However, due to the properties of T, the cardinality of any model is greater than or equal to κ. On the other hand, by the downward theorem, since T has a model with cardinality $\leq \kappa$, it follows that there exists a model with exact cardinality κ. □

Conservative extension

Definition

Let T and T' be theories in languages $\mathcal{L}$ and $\mathcal{L}'$, respectively, and $\mathcal{L} \subset \mathcal{L}'$. Then, T' is called a **conservative extension** of T if for any sentence σ in $\mathcal{L}$, $T \vdash \sigma \ \Leftrightarrow\ T' \vdash \sigma$.

Theorem

If an $\mathcal{L}$-theory $T \vdash \forall x_1 \cdots \forall x_n \exists y \varphi(x_1, \ldots, x_n, y)$, then $T' := T \cup \{\forall x_1 \cdots \forall x_n \varphi(x_1, \ldots, x_n, \mathtt{f}(x_1, \ldots, x_n))\}$ in $\mathcal{L} \cup \{\mathtt{f}\}$ is a conservative extension of T.

Proof

- Suppose $T \vdash \forall x_1 \cdots \forall x_n \exists y\ \varphi(x_1, \ldots, x_n, y)$. Let $\mathfrak{A}$ be any model of T. By axiom of choice, we construct a function $\mathtt{f}^{\mathfrak{A}}$ on $\mathfrak{A}$ such that

$$\mathfrak{A} \models \forall x_1 \cdots \forall x_n\ \varphi(x_1, \ldots, x_n, \mathtt{f}(x_1, \ldots, x_n))$$

- Then $\mathfrak{A}^* \equiv \mathfrak{A} \cup \{\mathtt{f}^{\mathfrak{A}}\}$ is a model of T'. Take any theorem σ of T' in the language $\mathcal{L}$. Though it is true in $\mathfrak{A}^*$, its truth value is irrelevant to $\mathtt{f}^{\mathfrak{A}}$. So, σ should hold in $\mathfrak{A}$.
- Since $\mathfrak{A}$ is an arbitrary model of T, by the completeness theorem we have $T \vdash \sigma$. □

If a formula φ is quantifier-free, $\forall x_1 \cdots \forall x_n \varphi$ is called a **$\forall$-formula** or **Π_1-formula**.

Theorem

Every theory T has a conservative extension theory T' consisting only of $\forall$-sentences.

Proof.

- For each formula $\exists y \varphi(x_1, \dots, x_n, y)$ in a language $\mathcal{L}$ with no free variables other than $x_1, \dots, x_n$, we add a new function symbol $\mathtt{f}_{\exists y \varphi(x_1,\dots,x_n,y)}$ and collect them as F_1. Put

$$S_1 = \{\forall x_1 \cdots \forall x_n (\exists y \varphi(x_1, \dots, x_n, y) \leftrightarrow \varphi(x_1, \dots, x_n, \mathtt{f}_{\exists y \varphi(x_1,\dots,x_n,y)}(x_1, \dots, x_n))) : \exists y \varphi(x_1, \dots, x_n, y) \text{ is a formula in } \mathcal{L}\}$$

  By the last theorem, for any theory T of $\mathcal{L}$, $T \cup S_1$ is a conservative extension of T.

- Next, for each formula of the form $\exists y \varphi(x_1, \dots, x_n, y)$ in the language $\mathcal{L} \cup F_1$, we add a new function symbol and collect them as F_2 and similarly define S_2.

- By repeating this process, we finally put

$$F = \bigcup_{i \in \mathbb{N}} F_i, \quad S = \bigcup_{i \in \mathbb{N}} S_i$$

- Then, for any $\mathcal{L}$-theory T, $T \cup S$ is a conservative extension of T, called an (**iterated**) **Skolem extension** of T. A symbol belonging to F is called a **Skolem function**.

- Under the Skolem axioms S, any formula φ in $\mathcal{L}' = \mathcal{L} \cup F$ is equivalent to a $\forall$-formula, which can be shown by induction on the construction of φ.

- Moreover, in order to prove that any formula is equivalent to a $\forall$-formula, we may restrict the Skolem axioms S to the following set.

$$S' = \{\forall x_1 \dots \forall x_n \forall y(\varphi(x_1, \dots, x_n, y) \to \varphi(x_1, \dots, x_n, \mathtt{f}_{\exists y \varphi(x_1,\dots,x_n,y)}(x_1, \dots, x_n))) :$$
$$\varphi(x_1, \dots, x_n, y) \text{ is a quantifier-free formula of } \mathcal{L}'\}$$

  Note here that all formulas in S' are $\forall$-sentences.

- Let us consider an example, we first transform a formula into prenex normal form by pushing an inner quantifier forward. For instance, change $\theta \land \forall x \xi(x)$ to $\forall z(\theta \land \xi(z))$ by replacing the bound variable x with a new variable z if necessary.

- Now take a formula $\exists x \forall y \exists z \theta(x, y, z)$ or $\exists x \neg \exists y \neg \exists z \theta(x, y, z)$ as an example. First replace z in $\theta(x, y, z)$ with $\mathtt{f}_{\exists z \theta(x,y,z)}(x, y) \in F_1$ and put the following into S_1

$$\forall x, y, z(\theta(x, y, z) \to \theta(x, y, \mathtt{f}_{\exists z \theta(x,y,z)}(x, y))).$$

- For simplicity, we write $\theta_1(x, y)$ for $\theta(x, y, \mathtt{f}_{\exists z \theta(x,y,z)}(x, y))$. Next, replace y in $\neg\theta_1(x, y)$ with $\mathtt{f}_{\exists y \neg \theta_1(x,y)}(x) \in F_2$ and put the following into S_2

$$\forall x, y(\neg\theta_1(x, y) \to \neg\theta_1(x, \mathtt{f}_{\exists y \neg \theta_1(x,y)}(x))).$$

- Again for simplicity, we write $\theta_2(x)$ for $\neg\theta_1(x, \mathtt{f}_{\exists y \neg\theta_1(x,y)}(x))$. Replace x in $\neg\theta_2(x)$ with a constant $\mathtt{f}_{\exists x \neg\theta_2(x)} \in F_3$ and put the following into S_3

$$\forall x(\neg\theta_2(x) \to \neg\theta_2(\mathtt{f}_{\exists x \neg\theta_2(x)})).$$

- Then under the assumption S_3, we have

$$\begin{aligned} \exists x \forall y \exists z \theta(x,y,z) &\leftrightarrow \exists x \forall y \theta_1(x,y) \\ &\leftrightarrow \exists x \neg \exists y \neg \theta_1(x,y) \\ &\leftrightarrow \exists x \neg\theta_2(x) \\ &\leftrightarrow \neg\theta_2(\mathtt{f}_{\exists x \neg\theta_2(x)}). \end{aligned}$$

  Thus, $\exists x \forall y \exists z \theta(x,y,z)$ is equivalent to a quantifier-free sentence.

- For each axiom (sentence) in the theory T, we rewrite it as a quantifier-free sentence in $\mathcal{L} \cup F$ and collect all of them as T''.

- Then $T' = T'' \cup S'$ is a conservative extension of T consisting of only $\forall$-sentences. $\square$

Next, we will consider an interpretation of a theory into a theory of a different language.

- First of all, we discuss a function symbol introduced by definition, which is a special case of interpretation as we will see later.

Assume $T \vdash \forall x_1 \cdots \forall x_n \exists! y \varphi(x_1, \ldots, x_n, y)$. Here, $\exists! y \psi(y)$ means "there exists a unique y that satisfies $\psi(y)$."

Then the theory $T' = T \cup \{\forall x_1 \cdots \forall x_n \forall y(\varphi(x_1, \ldots, x_n, y) \leftrightarrow \mathtt{f}(x_1, \ldots, x_n) = y)\}$ is called an **expansion** of T **by definition**. T' is a conservative extension of T.

Given a formula ψ of $\mathcal{L} \cup \{\mathtt{f}\}$, we construct $\psi^{-\mathtt{f}}$ in $\mathcal{L}$ by the following procedure.

(1) If ψ does not include $\mathtt{f}$, then terminate this process by setting $\psi^{-\mathtt{f}} = \psi$.

(2) If ψ contains $\mathtt{f}$, take an atomic subformula θ containing it, and choose a subterm $\mathtt{f}(t_0, \ldots, t_{n-1})$ in it such that no t_i contains $\mathtt{f}$.

(3) In θ, replace the subterm selected in (2) with a new variable y and call it $\theta_1(y)$.

(4) Replace θ in ψ by $\exists y(\varphi(t_0, \ldots, t_{n-1}, y) \wedge \theta_1(y))$, and then we regard it as a new ψ, and then go to (1).

It is easy to see that $T' \vdash \psi \leftrightarrow \psi^{-\mathtt{f}}$

Lemma

Let θ be a subformula of φ. Assume $T \vdash \theta \leftrightarrow \theta'$. Let φ' be a formula obtained from φ by replacing some (or all) occurrences of θ in φ with θ'. Then $T \vdash \varphi \leftrightarrow \varphi'$.

Proof. By the completeness theorem, it is enough to show that in any model $\mathfrak{A}$ of T, φ and φ' have the same truth value. This is obvious from Tarski's truth definition clauses. □

Relational expansion

- Expand a theory by a new relational symbol R as follows:

$$T' = T \cup \{\forall x_1 \cdots \forall x_n(\varphi(x_1, \dots, x_n) \leftrightarrow \mathrm{R}(x_1, \dots, x_n))\},$$

  It is also a conservative extension of T.

- Let $\psi^{-\mathrm{R}}$ denote a formula obtained from ψ by replacing all occurrences of $\mathrm{R}(t_1, \dots, t_n)$ with $\varphi(t_1, \dots, t_n)$. Then

$$T' \vdash \psi \leftrightarrow \psi^{-\mathrm{R}}$$

Homework (difficult)

Let Σ be a theory in a language $\mathcal{L}$ including an n-ary relation symbol R and some others. Then, R is said to be explicitly definable in Σ, if there exists a formula $\varphi(x_0, \ldots, x_{n-1})$ in $\mathcal{L} - \{R\}$ such that

$$\Sigma \vdash \forall x_0, \ldots, x_{n-1}(R(x_0, \ldots, x_{n-1}) \leftrightarrow \varphi(x_0, \ldots, x_{n-1})).$$

Now, we construct Σ' from Σ by replacing all occurrences of R by a new symbol R'. Then, R is said to be implicitly definable in Σ, if the following hold

$$\Sigma \cup \Sigma' \vdash \forall x_0, \ldots, x_{n-1}(R(x_0, \ldots, x_{n-1}) \leftrightarrow R'(x_0, \ldots, x_{n-1})).$$

Show that R is explicitly definable in Σ iff R is implicitly definable in Σ.

Now we are ready to define a language interpretation.

Definition

Given two languages $\mathcal{L}, \mathcal{L}'$ and a theory T' in the language $\mathcal{L}'$. A pair $\langle U, I\rangle$ that satisfies the following is called a **interpretation (translation)** of language $\mathcal{L}$ (in T').

(1) U is a one-variable formula in $\mathcal{L}'$. (It represents the domain of the theory in $\mathcal{L}$.)

(2) I is a function from $\mathcal{L}$ to formulas in $\mathcal{L}'$, and if $\mathtt{f}$ is a n-ary function symbol, $I(\mathtt{f})$ is an $(n+1)$-ary formula; if R is an n-ary relation symbol, $I(\mathrm{R})$ is also an n-ary formula.

(3) $T' \vdash \exists x U(x)$.

(4) For each functional symbol f,

$$T' \vdash \forall x_1 \cdots \forall x_n (U(x_1) \wedge \cdots \wedge U(x_n) \to \exists! y (I(\mathtt{f})(x_1, \ldots, x_n, y) \wedge U(y))).$$

- Next, we want to interpret the $\mathcal{L}$-formulas.
- However, we should notice that $\forall x_1 \cdots \forall x_n \exists! y I(\mathtt{f})(x_1, \ldots, x_n, y)$ may not hold outside of U. So, we take a new constant a and modify $I(\mathtt{f})$ as follows:

$$I'(\mathtt{f})(x_1, \ldots, x_n, y) \Leftrightarrow$$

$$((U(x_1) \wedge \cdots \wedge U(x_n)) \wedge I(\mathtt{f})(x_1, \ldots, x_n, y)) \vee ((\neg U(x_1) \vee \cdots \vee \neg U(x_n)) \wedge y = a).$$

Then, let f be a function symbol defined by $I'(\mathtt{f})$.

- Also, let R be a relational symbol defined by $I(\mathrm{R})$. Then, after interpretation, the terms of $\mathcal{L}$ will remain unchanged, and so will the atomic formulas and the propositional connectives.
- We only need to deal with quantifiers. If we denote the interpretation of φ in $\mathcal{L}$ by φ^I,
  (1) $(\exists x \psi)^I$ is $\exists x(U(x) \wedge \psi^I)$.
  (2) $(\forall x \psi)^I$ is $\forall x(U(x) \rightarrow \psi^I)$.

Definition

- Let T and T' be theories of languages $\mathcal{L}$ and $\mathcal{L}'$, respectively. Suppose that $\langle U, I\rangle$ is an interpretation of language $\mathcal{L}$ in T'.
- Then, $\langle U, I\rangle$ is said to be the **interpretation** of the theory T in T', if for any sentence σ in $\mathcal{L}$,

$$T \vdash \sigma \quad \Rightarrow \quad T' \vdash \sigma^I.$$

- If there is an interpretation of T in T', T is said to be **interpretable** within T'.
- Moreover, if the following holds

$$T \vdash \sigma \quad \Leftrightarrow \quad T' \vdash \sigma^I$$

$\langle U, I\rangle$ is called a **faithful interpretation** of T' in T.

Example 7

If T is an expansion of T' by defition, then there is a faithful interpretation $\langle U, I\rangle$ of T in T'. Let $U(x)$ be $x = x$. For a defined function $\mathtt{f}$ and relation R, let $I(\mathtt{f})$ and $I(\mathrm{R})$ be their definitions. The interpretations of other symbols are the same as the originals.

Example 8

Let $\mathfrak{N} = (\mathbb{N}, +, \cdot, 0, 1, <)$, $\mathfrak{Z} = (\mathbb{Z}, +, \cdot, 0, 1)$.
There exists a faithful interpretation $\langle U, I\rangle$ from Th($\mathfrak{N}$) to Th($\mathfrak{Z}$):

$$U(x) \equiv \exists x_1 \exists x_2 \exists x_3 \exists x_4 (x = x_1 \cdot x_1 + \cdots + x_4 \cdot x_4)$$

$$I(+)(l, m, n) \equiv l + m = n, \quad I(\cdot)(l, m, n) \equiv l \cdot m = n$$

$$I(0)(n) \equiv n = 0, \quad I(1)(n) \equiv n = 1$$

$$I(<)(m, n) \equiv \exists x (U(x) \wedge x \neq 0 \wedge m + x = n)$$

Problem 4

Show that there exists a faithful translation $\langle U, I\rangle$ from Th($\mathfrak{Z}$) to Th($\mathfrak{N}$).

Problem 5

1. Show that Peano arithmetic PA is interpretable within ZF set theory.
2. Show that ZF without Infinity axiom is interpretable within PA.

- If a faithful translation from T to T' exists, provability in T is reducible to that of T'. Therefore, if T' is decidable, so is T.
- Conversely, to show the undecidability of T', it suffices to interpret an undecidable theory into T'.

Second order logic

- In first-order logic, quantifiers $\forall$ and $\exists$ range over the elements of a structure.
- Second-order logic enables us to use quantifiers over relations and functions on the domain. For simplicity, we deal only with quantification over relations, not functions.

Definition

Let $\varphi(\mathrm{R})$ be a first-order formula in language $\mathcal{L} \cup \{\mathrm{R}\}$. The truth values of second order logic formulas $\forall R\varphi(R)$ and $\exists R\varphi(R)$ in a $\mathcal{L}$-structure $\mathcal{A}$ is defined as follows.

$$\mathcal{A} \models \forall R\varphi(R) \Leftrightarrow \text{for any } \dot{\mathrm{R}} \subseteq A^n, (\mathcal{A}, \dot{\mathrm{R}}) \models \varphi(\mathrm{R}) \text{ holds.}$$

$$\mathcal{A} \models \exists R\varphi(R) \Leftrightarrow \text{there exists } \dot{\mathrm{R}} \subseteq A^n \text{ such that } (\mathcal{A}, \dot{\mathrm{R}}) \models \varphi(\mathrm{R}).$$

- In the following, we do not strictly distinguish among the relation variable R, relation $\dot{\mathrm{R}}$, and relation constant (symbol) R.
- The concepts of free and bound variables can be introduced for second-order formulas as those in first-order formulas.

- The critical issue is how to consider the domain of second-order variables.
- In the above definition, we use "any $\dot{\mathrm{R}} \subseteq A^n$" to mean that "all" subsets of A^n should be considered. A structure with such an interpretation is called a **standard structure** of second-order logic, although this cannot be formally defined.
- For simplicity, we restrict second-order variables to unary relations, namely subsets of the first-order domain. This is called **monadic second-order logic** (MSO).

Theorem (Gödel)

The validity of (M)SO in standard structures is not axiomatizable.

Proof.

- Assume MSO were axiomatized. We can define second-order Peano Arithmetic PA_2 by adding axioms of arithmetic such as PA to MSO.
- In any model $\mathcal{M}$ of PA_2, since all subsets of the first-order domain M belong to the second-order domain, then the smallest set N containing 0 and closed under $+1$ also belongs to the second-order domain. Here, N is isomorphic to the standard $\mathbb{N}$.
- Assuming PA_2 includes mathematical induction, N must coincide with the whole M. In other words, M is isomorphic to $\mathbb{N}$, and so any model of PA_2 is isomorphic to $\mathbb{N} \cup \mathcal{P}(\mathbb{N})$. Therefore, there is no sentence independent from PA_2. This condradicts with Gödel's first incompleteness theorem. □

- Instead, L. Henkin considered a **general structure** of second-order logic, whose second-order part varies similarly to the first-order logic domain. In other words, such a logic can be regarded as two-sorted first-order logic.
- We only define the general structure of monadic second-order logic. The monadic second-order variables (also called **set variables**) are denoted by $X, Y, Z, \ldots$, and the atomic formula $X(t)$ is also written as $t \in X$.

Definition

A **general structure** of monadic second-order logic $\mathcal{B} = (\mathcal{A}, \mathcal{S})$ consists of first-order logic structure $\mathcal{A}$ and set $\mathcal{S} \subset \mathcal{P}(A)$. The set quantifiers range over $\mathcal{B}$ as follows.

$$\mathcal{B} \models \forall X \varphi(X) \Leftrightarrow \text{for any } S \in \mathcal{S}, \mathcal{B} \models \varphi(S) \text{ holds,}$$

$$\mathcal{B} \models \exists X \varphi(X) \Leftrightarrow \text{there exists } S \in \mathcal{S} \text{ such that } \mathcal{B} \models \varphi(S).$$

- A general structure can also be viewed as a first-order structure with two domains (A and $\mathcal{S}$) (or split into two domains). The formalization such as a derivation system is almost the same as first-order logic, just by preparing two kinds of variables.

- Henkin assumed that the general structure should satisfy certain amounts of comprehension axiom and axiom of choice. The comprehension axiom is an assertion that for a formula $\varphi(x)$ with no free occurrence of X, $\exists X \forall x(x \in X \leftrightarrow \varphi(x))$, that is, the set $\{x : \varphi(x)\}$ exists in the second-order domain, where $\varphi(x)$ does not include the variable X.

Theorem (Henkin's completeness theorem of MSO)

An MSO formula is provable from appropriate comprehension and other axioms in two-sorted first-order system if and only if it is true in any general structure that satisfies those axioms.

- This theorem can be proved in the same way as in first-order logic.
- It can also be generalized to higher-order logics. In fact, Henkin's proof for the completeness theorem of first-order logic was made with such a generalization scheme.

Example 1, 2: MSO is more expressive than first-order logic FO

FO cannot distinguish $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$. In MSO, it can express that "a bounded set $X(\neq \varnothing)$ has a least upper bound", and hence $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$ are distinguishable.

MSO can express the sentence that determines the parity (even or odd) of the length of a finite linear order, which is not expressible by FO.

Example 3: SO is more expressive than MSO

The MSO theory of $(\mathbb{N}, x+1, 0)$ is decidable due to Büchi. But SO theory of $(\mathbb{N}, x+1, 0)$ is not, since addition $m + n = k$ is defined by

$$\forall R([R(0, m) \wedge \forall x, y(R(x, y) \to R(x + 1, y + 1))] \to R(n, k),$$

and multiplication can be defined in a similar way, which means that first-order arithmetic is embedded into the theory.

Thank you for your attention!