Logic and Foundations I

Part 1. Equational theory

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BIMSA

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Outline of the Course

1. This is an introductory graduate-level course in **mathematical logic**. In this semester, we discuss more advanced topics emphasizing on **foundations of mathematic**.
2. Every Thursday, we give a lecture three (strictly 2.5) hours long including a problem session. We will also assign simple homework problems or questionnaires to registered students, who are motivated to attend the class continuously. Normally, homeworks are due next Monday.
3. TA (Dr. Li) will handle homeworks as well as questions and comments from students via WeChat. We may assign harder problems to students, who will presumably go to the research level with us in the following years.
4. Lecture slides will be uploaded on the lecture page at BIMSA.

Education

- ⋆ Tokyo Institute of Technology
  Information Science, Bachelor, Master
- ⋆ University of California, Berkeley
  Mathematics,
  Ph.D. (Advisor: Leo Harrington)

Introducing myself

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Teaching Jobs

- ⋆ 1986 ∼ 1991, Tokyo Institute of Technology
  Assistant Professor, Dept. of Info. Sci.;
  Visiting PennState.
- ⋆ 1991 ∼ 1997, Tohoku University
  Associate Professor, Dept. of Math.;
  Visiting Oxford.
- ⋆ 1997 ∼ 2022, Tohoku University
  Professor (2021 , Emeritus), Mathematical Institute and Research Alliance Center for Mathematical Sciences.
- ⋆ 2022 ∼ now, BIMSA, Professor (Research Fellow).

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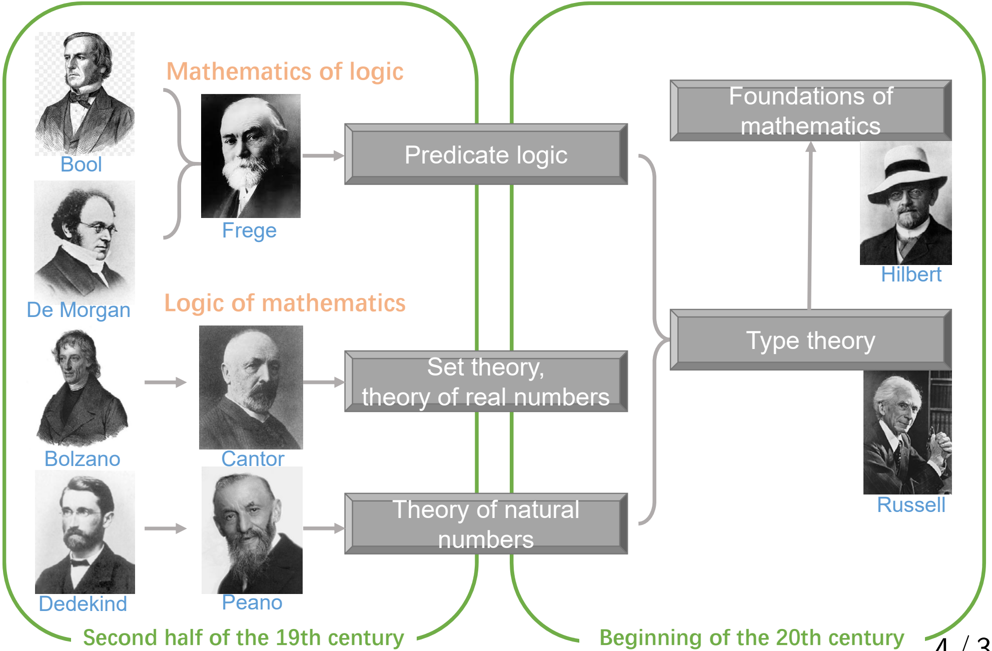
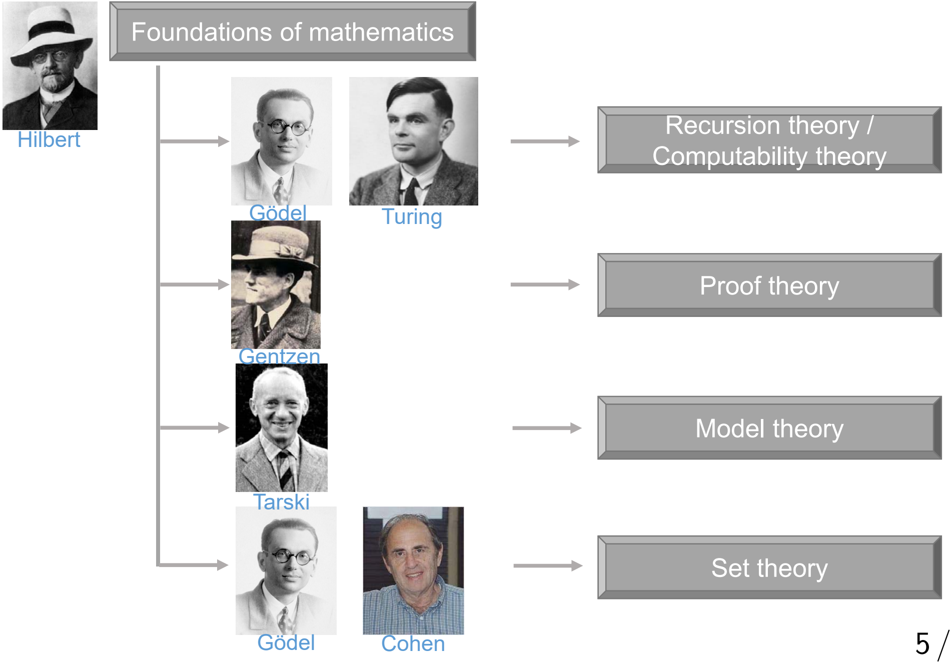
Introducing myself

Speciality

Mathematical logic, especially definabilty and computability theory. Among others, I have contributed to second-order arithmetic and reverse mathematics, and supervised fifteen doctoral students in this area.
See https://sendailogic.com/tanaka/.


Mathematics of logic
Bool
Frege
Predicate logic
De Morgan
Logic of mathematics
Bolzano
Cantor
Set theory,
theory of real numbers
Dedekind
Peano
Theory of natural numbers
Foundations of mathematics
Hilbert
Type theory
Russell
Second half of the 19th century
Beginning of the 20th century


Foundations of mathematics
Hilbert
Gödel
Turing
Recursion theory / Computability theory
Gentzen
Proof theory
Tarski
Model theory
Gödel
Cohen
Set theory

Logic and Foundations I

- **Part 1. Equational theory**
- **Part 2. First order theory**
- **Part 3. Model theory**
- **Part 4. First order arithmetic and incompleteness theorems**

Part 1. Schedule

- Sep. 21, (1) Formal systems of equation
- Sep. 28, (2) Free algebras and Birkhoff's theorem
- Oct. 12, (3) Boolean algebras
- Oct. 19, (4) Computable functions and general recursive functions

Today's topics

1. Equational Theory
2. Algebraic structures

Introduction to Equation Theory

- In part 1, we study mathematical theories that are characterized by axioms in the form of equations $(=)$.
- The class of models of such a theory has some intriguing properties (e.g., it is closed under the Cartesian product).
- The Birkhoff completeness theorem and the equational class theorems are two major results, which will also be extended to more general theories involving logical symbols in the following parts of this course.
- In this part, we also discuss Boolean algebra as an important equational theory. At last, we introduce general recursive functions as another application of equational theory.

- To begin with, we look at an equational theory of groups in order to observe how a mathematician proves. Among various ways to describe the theory of groups, we adopt the following equational axioms.

Definition

Group theory $\mathrm{G_p}$ consists of the following three axioms.

$$\begin{array}{llcll} \text{G1}: & (x \cdot y) \cdot z & = & x \cdot (y \cdot z) & \text{(associativity)} \\ \text{G2}: & \mathsf{e} \cdot x & = & x & \text{(left identity)} \\ \text{G3}: & x^{-1} \cdot x & = & \mathsf{e} & \text{(left inverse)} \end{array}$$

where x, y and z are variables, e is a constant, and $^{-1}$ represents a unary function.

- Consider a structure $\mathfrak{G} = (\mathbf{G}, *, ^{\sim}, e)$, where $\mathbf{G}$ is a non-empty set, $*$ a binary function, $^{\sim}$ a unary function on $\mathbf{G}$, and e an element of $\mathbf{G}$.
- $\mathfrak{G}$ is called a **model** of $\mathrm{G_p}$, or simply **group**, if by interpreting the symbols $\cdot$, $^{-1}$, e of $\mathrm{G_p}$ as $*$, $^{\sim}$, e on $\mathbf{G}$, the three equalities hold for any assignment of an element of $\mathbf{G}$ to each variable.

- In general, when a sentence σ holds in a structure $\mathfrak{M}$ (in other words, σ is true in $\mathfrak{M}$), we write $\mathfrak{M} \models \sigma$.
- A set T of sentences (logical formulas or axioms) is called a **theory**. If all the sentences $\sigma \in T$ hold in $\mathfrak{M}$, we say that $\mathfrak{M}$ is a **model** of T or T has a model $\mathfrak{M}$, denote $\mathfrak{M} \models T$. Here $\models$ is read as "double turnstile".
- If a sentence σ holds for all models of a theory T, σ is called a **consequence** of T or σ is **valid** in T, written as $T \models \sigma$
- We take a look at the following theorem and proof, as an example of an argument for a consequence of group theory $\mathrm{G_p}$.

Theorem (1)

$\mathrm{G_p} \models x \cdot x^{-1} = \mathsf{e}$ [right inverse]

Proof.

Let $\mathfrak{G} = (\mathbf{G}, *, {}^\sim, e)$ be an arbitrary group. Pick any element a of $\mathbf{G}$. We claim $a * a^\sim = e$.
First, we show $a = e$ if $a * a = a$. Multiply by $a^\sim$, on both sides of $a * a = a$, we have $a^\sim * (a * a) = a^\sim * a$.
The left-hand side of this is

$$\begin{aligned} a^\sim * (a * a) &= (a^\sim * a) * a && \text{(by G1)} \\ &= e * a && \text{(by G3)} \\ &= a && \text{(by G2)} \end{aligned}$$

Definition (revisited)

Group theory $\mathrm{G_p}$ consists of the following three axioms.

$$\begin{aligned} \mathrm{G1}: \quad (x \cdot y) \cdot z &= x \cdot (y \cdot z) \\ \mathrm{G2}: \quad \mathsf{e} \cdot x &= x \\ \mathrm{G3}: \quad x^{-1} \cdot x &= \mathsf{e} \end{aligned}$$

Since the right-hand side $a^\sim * a$ is equal to e from G3, we obtain $a = e$.
Now, we have

$$\begin{aligned} (a * a^\sim) * (a * a^\sim) &= a * (a^\sim * (a * a^\sim)) && \text{(by G1)} \\ &= a * ((a^\sim * a) * a^\sim) && \text{(by G1)} \\ &= a * (e * a^\sim) && \text{(by G3)} \\ &= a * a^\sim && \text{(by G2)} \end{aligned}$$

Hence, $a * a^\sim = e$. □

Problem 1

$\mathrm{G_p} \models x \cdot \mathsf{e} = x$ [right identity].

Problem 2

Let $\mathrm{G'_p}$ be a theory obtained by replacing G3 [left inverse] in $\mathrm{G_p}$ with

$$x \cdot x^{-1} = \mathsf{e} \qquad \text{[right inverse]}.$$

Prove that G3 does not hold in $\mathrm{G'_p}$, i.e.,

$$\mathrm{G'_p} \not\models x^{-1} \cdot x = \mathsf{e}.$$

- The proof of the above Theorem in Page 10 is an ordinary argument in mathematics.
- However, if you think twice, it is not at all obvious to take an arbitrary group $\mathbf{G}$ and select an arbitrary element a to discuss it.
- Can we say that the claim is true if there exists a right inverse in the groups one can imagine or people have found until today?
- However, we should notice that once we have fixed a group $\mathbf{G}$ and its arbitrary element a, the rest is a simple transformation of formulas.
- The transformation is obtained by starting from the axioms of $\mathrm{G_p}$ and applying the rules of equality.
- Indeed, it doesn't really matter which group you would choose to handle. Any structure that satisfies the axioms will work out even if you cannot imagine it.

Formal system of an equational theory

- Let us now introduce a formal system of equations. We will give the general definition of a language, a term, etc. later.
- Here, we may consider a **language** as a set of mathematical symbols such as $\cdot$, $^{-1}$, and e in group theory.
- A string consisting of these symbols and variables with parentheses, is called a **term** if it is properly combined to denote an element of a given structure.
- Then, for two terms s, t, the symbol string $s = t$ is called an **equation**.

The deductive system, which derives the consequences of a theory T, is defined as follows.

- The axiom (2) $t = t$ expresses the reflexivity of equality.
- $\dfrac{s = t}{t = s}$ (sym) expresses the symmetricity rule.
- $\dfrac{s = t \quad t = u}{s = u}$ (trans) expresses the transitivity rule.
- $\dfrac{s(x) = t(x)}{s(u) = t(u)}$ (sub) is a substitution rule for replacing all occurrences of variable x with a term u.
- $\dfrac{s_1 = t_1 \ \ \dots \ \ s_n = t_n}{\mathtt{f}(s_1, \dots, s_n) = \mathtt{f}(t_1, \dots, t_n)}$ (comp) guarantees that equality is preserved by function composition.
  E.g, in group theory, $s_1 = t_1, s_2 = t_2 \Rightarrow s_1 \cdot s_2 = t_1 \cdot t_2$, and $s = t \Rightarrow s^{-\mathbf{1}} = t^{-\mathbf{1}}$.

Definition

A **proof tree** (or **proof**) in equational theory T is defined inductively as follows:

1. An equation as an axiom is a proof tree by itself.
2. If P_i is a proof tree for $s_i = t_i$ $(1 \leq i \leq n)$, and

$$\frac{s_1 = t_1 \quad \dots \quad s_n = t_n}{s = t}$$

is an inference rule, then

$$\frac{P_1 \quad \dots \quad P_n}{s = t}$$

is a proof tree for $s = t$.

If $s = t$ has a proof tree in T, we write

$$T \vdash s = t,$$

where $\vdash$ is read as "turnstile".

We work in equational theory $\mathrm{G_p}$ unless otherwise noted. We write xy in short for $x \cdot y$.

Example 1-1: A proof tree P_1 for $(x^{-1}x)x^{-1} = x^{-1}$

Let P_1 be the following proof tree for $(x^{-1}x)x^{-1} = x^{-1}$,

$$\dfrac{\dfrac{x^{-1}x = \mathsf{e} \quad x^{-1} = x^{-1}}{(x^{-1}x)x^{-1} = \mathsf{e}x^{-1}}\ (\mathrm{comp}) \qquad \dfrac{\mathsf{e}x = x}{\mathsf{e}x^{-1} = x^{-1}}\ (\mathrm{sub})}{(x^{-1}x)x^{-1} = x^{-1}}\ (\mathrm{trans}).$$

Example 1-2: A proof tree P_2 for $x^{-1}(xx^{-1}) = (x^{-1}x)x^{-1}$

$$\dfrac{\dfrac{(xy)z = x(yz)}{(x^{-1}x)x^{-1} = x^{-1}(xx^{-1})}\ (\mathrm{sub} \times 3\mathrm{times})}{x^{-1}(xx^{-1}) = (x^{-1}x)x^{-1}}\ (\mathrm{sym})$$

Example 1-3: A proof tree for $(xx^{-1})(xx^{-1}) = (xx^{-1})$

Let P_3 be a proof tree for $(xx^{-1})(xx^{-1}) = x(x^{-1}(xx^{-1}))$. The following is a proof tree for $(xx^{-1})(xx^{-1}) = (xx^{-1})$.

$$\dfrac{P_3 \qquad \dfrac{x = x \qquad \dfrac{P_2 \quad P_1}{x^{-1}(xx^{-1}) = x^{-1}}\,(\mathrm{trans})}{x(x^{-1}(xx^{-1})) = xx^{-1}}\,(\mathrm{comp})}{(xx^{-1})(xx^{-1}) = (xx^{-1})}\,(\mathrm{trans})$$

Problem 3

Using the examples above, construct a proof tree for $\mathrm{G_p} \vdash xx^{-1} = \mathsf{e}$.

Problem 4

Construct a proof tree for $\mathrm{G_p} \vdash x\mathsf{e} = x$.

Exercise

Consider an equation system consisting of binary operator symbol $\cdot$ and c, d, e as constants.

1. For a theory $T = \{c \cdot x = x, x \cdot d = x\}$, construct a proof tree for $T \vdash c = d$.
2. For a theory $T' = \{c \cdot x = x, e \cdot x = x, x \cdot d = x\}$, construct a proof tree for $T' \vdash c = e$.
3. Prove that $c = e$ does not hold in some model of $T'' = \{(x \cdot y) \cdot z = x \cdot (y \cdot z), c \cdot x = x, e \cdot x = x\}$.

Birkhoff's completeness theorem

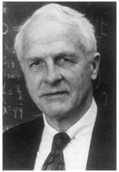
- For an equational theory T, the following relationship holds.

Birkhoff's completeness theorem (1935)

$$T \models s = t \Leftrightarrow T \vdash s = t.$$

In other words, "$s = t$ is a consequece of theory T ($T \models s = t$)" can be completely captured by a finite diagram of a proof tree for $s = t$.

- It can be regarded as a special case of **Gödel's completeness theorem** (1930), which asserts that mathematical arguments in first-order logic can be completely formalized.


Garrett Birkhoff


Kurt Gödel

Birkhoff's completeness theorem

- We will prove Birkhoff's completeness theorem next time, but we briefly explain the idea of the proof.
- $T \models s = t \Leftarrow T \vdash s = t$ (the soundness of T) is easier. Let $\mathfrak{M}$ be any model of T. Then we can show by induction that all equations appearing in a proof tree for $T \vdash s = t$ holds in $\mathfrak{M}$. Especially the bottom $s = t$ holds in $\mathfrak{M}$.
- However, $\Rightarrow$ is not easy. To show the contrapositive, we first assume $T \nvdash s = t$, and construct a structure $\mathfrak{M}$ such that

$$\mathfrak{M} \models T \text{ and } \mathfrak{M} \not\models s = t.$$

Such a structure is obtained as the "free algebra" generated by the variables appearing in s, t. Before introducing it, we first review the basic concepts of general algebra.

Algebraic language

Definition

An **algebraic language** is a list of function symbols

$$\mathcal{L} = (\mathtt{f}_0, \mathtt{f}_1, \dots),$$

where each $\mathtt{f}_i$ is associated with a natural number m_i, called its **arity**, that is, $\mathtt{f}_i$ stands for an m_i-ary function. A 0-ary function symbol is also regarded as a **constant**.
$\rho = (m_0, m_1, \dots)$ is called the **similarity type** of $\mathcal{L}$.

Note that there may be infinitely many (especially uncountably many) symbols in an algebraic language $\mathcal{L}$.

Algebraic structures

Definition

- An **algebraic structure** $\mathfrak{A}$ in a language $\mathcal{L} = (\mathtt{f}_0, \mathtt{f}_1, \dots)$ (or simply **$\mathcal{L}$-algebra**) consists of a non-empty set A and a list of m_i-ary functions $\mathtt{f}_i^{\mathfrak{A}} : A^{m_i} \to A$, that is,

$$\mathfrak{A} = (A, \mathtt{f}_0^{\mathfrak{A}}, \mathtt{f}_1^{\mathfrak{A}}, \dots).$$

- We say that A is the **domain** or **universe** of $\mathfrak{A}$, denoted by $|\mathfrak{A}|$.
- For a 0-ary function symbol or constant c, $\mathrm{c}^{\mathfrak{A}}$ is an element of A.

Example

- A group $\mathfrak{G}$ is an algebraic structure $(\mathbf{G}, *, {}^{\sim}, e)$ in an algebraic language $\mathcal{L} = (\cdot, {}^{-1}, \mathrm{e})$.
- Then $\cdot^{\mathfrak{G}}$ is a binary function $*$, $({}^{-1})^{\mathfrak{G}}$ is a unary function ${}^{\sim}$, and $\mathrm{e}^{\mathfrak{G}}$ is a 0-ary function e. Therefore, the similarity type of $\mathcal{L}$ is $(2, 1, 0)$.

Terms

Definition

- In particular, constants (0-ary function symbols) are terms.
- We may write a term t including some variables (e.g., x, y) as $t(x, y)$.
- The term obtained from a term $t(x)$ by replacing all variables x appearing in it with a term s is expressed as $t(s)$.
- By convention, a binary function $\mathtt{f}(x, y)$ is also expressed as $x\mathtt{f}y$, e.g., $+(x, y)$ is written as $x + y$.

Homomorphism and isomorphism

We fix an algebraic language $\mathcal{L}$. We will only consider algebraic structures in this language unless otherwise stated, .

Definition

Let $\mathfrak{A}$, $\mathfrak{B}$ be $\mathcal{L}$-algebras. A morphism $\phi : \mathfrak{A} \to \mathfrak{B}$ is a **homomorphism** if for each n-ary function symbol $\mathtt{f}$ in $\mathcal{L}$, and for any $a_0, \ldots, a_{n-1} \in |\mathfrak{A}|$,

$$\phi(\mathtt{f}^{\mathfrak{A}}(a_0, \ldots, a_{n-1})) = \mathtt{f}^{\mathfrak{B}}(\phi(a_0), \ldots, \phi(a_{n-1})).$$

Moreover, ϕ is said to be an **isomorphism** if ϕ is bijective.
Then we say that $\mathfrak{A}$ and $\mathfrak{B}$ are **isomorphic**, denoted by

$$\mathfrak{A} \cong \mathfrak{B},$$

if there exists an isomorphism $\phi : \mathfrak{A} \to \mathfrak{B}$.

Example 3

Consider a group $\mathfrak{A} = (\mathbb{Z}, +, -, 0)$ and a group $\mathfrak{B} = (\mathbb{R}^+, \cdot, 1/x, 1)$ where $\mathbb{R}^+ = \{r \in \mathbb{R} \mid r > 0\}$. Then there is a homomorphism $\phi : \mathfrak{A} \to \mathfrak{B}$ defined by

$$\phi(n) = 2^n.$$

In particular, we have $\phi(m+n) = \phi(m) \cdot \phi(n)$.
Furthermore, $\mathfrak{M} = (M, \cdot, 1/x, 1)$ with $M = \{2^n : n \in \mathbb{Z}\}$ is also a group, and we have $\mathfrak{A} \cong \mathfrak{M}$.

Problem 5

Prove that $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic iff there exist two homomorphisms $\phi : \mathfrak{A} \to \mathfrak{B}$ and $\psi : \mathfrak{B} \to \mathfrak{A}$ such that their composite functions $\psi \circ \phi$ and $\phi \circ \psi$ are both identity maps (id_A and id_B).

Definition

Let $\mathfrak{A}$ be an algebra and $\equiv$ be a binary relation on $|\mathfrak{A}|$. Then we say that $\equiv$ is a **congruence relation** on $\mathfrak{A}$ if $\equiv$ is an equivalence relation on $|A|$ (i.e., satisfying reflexivity, transitivity, and symmetricity) and for every functional symbol $\mathtt{f} \in \mathcal{L}$, and for any $a_0, \cdots, a_{n-1}, b_0, \cdots, b_{n-1} \in A$, we have

$$a_0 \equiv b_0, \ldots, a_{n-1} \equiv b_{n-1} \Rightarrow \mathtt{f}^{\mathfrak{A}}(a_0, \ldots, a_{n-1}) \equiv \mathtt{f}^{\mathfrak{A}}(b_0, \ldots, b_{n-1}).$$

Example 4

Let $\mathfrak{Z} = (\mathbb{Z}, +, \cdot, -, 0, 1)$ be a ring of integers. We define $m \equiv_3 n \Leftrightarrow$ "$m - n$ is a multiple of 3". To prove $\equiv_3$ is a congruence relation, we will show that if $m \equiv_3 n$ and $m' \equiv_3 n'$, then $m + m' \equiv_3 n + n'$ and $m \cdot m' \equiv_3 n \cdot n'$ and $-m \equiv_3 -n$.

Problem 6

Let $\mathfrak{H}$ be a normal subgroup of the group $\mathfrak{G}$ (for any $g \in |\mathfrak{G}|$ and $h \in |\mathfrak{H}|$, $ghg^{\sim} \in |\mathfrak{H}|$). Let $g_1 \equiv_H g_2 \Leftrightarrow g_1 g_2{}^{\sim} \in |\mathfrak{H}|$. Show that $\equiv_H$ is a congruence relation.

- Given an equivalence relation $\equiv$ on a set A, we call
$$\{x \in A : x \equiv a\}$$
the **equivalent class** or **residue class** of a $(\in A)$, denoted by $[a]$. A class of all equivalence classes is denoted by $A/\!\equiv$.
- Then, we can make an algebra $\mathfrak{A}/\!\equiv$ with domain $|\mathfrak{A}|/\!\equiv$ as follows.

Definition

Give a congruence relation $\equiv$ on an $\mathcal{L}$-algebra $\mathfrak{A}$. Each n-ary function symbol $\mathtt{f}$ in $\mathcal{L}$ is interpretated on $|\mathfrak{A}|/\!\equiv$ as follows: for all $a_0, \ldots, a_{n-1} \in |\mathfrak{A}|$,
$$\mathtt{f}^{\mathfrak{A}/\equiv}([a_0], \ldots, [a_{n-1}]) = [\mathtt{f}^{\mathfrak{A}}(a_0, \ldots, a_{n-1})].$$
The algebra $\mathfrak{A}/\!\equiv$ thus defined is called the **factor algebra** or **quotient algebra** of $\mathfrak{A}$ by $\equiv$.

In the above definition, the value of $\mathtt{f}^{\mathfrak{A}/\equiv}$ is (uniquely) determined by the fact that $\equiv$ is the congruence relation. That is, if $[a_0] = [a'_0], \ldots, [a_{n-1}] = [a'_{n-1}]$, then we have
$$[\mathtt{f}^{\mathfrak{A}}(a_0, \ldots, a_{n-1})] = [\mathtt{f}^{\mathfrak{A}}(a'_0, \ldots, a'_{n-1})].$$

Example 5

Consider the congruence relation in Example 3.
To make $\mathfrak{Z}/\equiv_3$, we first have $\mathbb{Z}/\equiv_3 = \{[0],[1],[2]\}$.
Then the operations on $\mathfrak{Z}/\equiv_3$ are defined as follows:
$[m] +^{\mathfrak{Z}/\equiv_3} [n] = [m+n]$, $[m] \cdot^{\mathfrak{Z}/\equiv_3} [n] = [m \cdot n]$, $-^{\mathfrak{Z}/\equiv_3}[m] = [-m]$, $0^{\mathfrak{Z}/\equiv_3} = [0]$, $1^{\mathfrak{Z}/\equiv_3} = [1]$.

Example 6

Let $\mathfrak{H}$ be a normal subgroup of the group $\mathfrak{G}$, then $\mathfrak{G}/\equiv_H$ is the usual residue group $\mathfrak{G}/\mathfrak{H}$. (see also Problem 6)

Lemma

If $\equiv$ *is a congruence relation on an algebra* $\mathfrak{A}$*, then* $\pi : \mathfrak{A} \to \mathfrak{A}/{\equiv}$ *defined by* $\pi(a) = [a]$ *is a homomorphism.*

By the definition of residue algebra, the proof of the lemma should be clear.
In the following, we represent the homomorphism π as $\pi_{\equiv}$.

Lemma

Let $\mathfrak{A}$, $\mathfrak{B}$ *be* $\mathcal{L}$*-algebras and* $\phi : \mathfrak{A} \to \mathfrak{B}$ *be a homomorphism. Now, we define a binary relation* $\equiv$ *on* A *as follows.*

$$a \equiv b \Leftrightarrow \phi(a) = \phi(b).$$

Then, $\equiv$ *is a congruence relation.*

Proof.
It is clear that $\equiv$ is an equivalence relation. To show the preservation property of $\mathtt{f}$, suppose $a_0 \equiv b_0, \ldots, a_{n-1} \equiv b_{n-1}$. Then,

$$\begin{array}{rcll} \phi(\mathtt{f}^{\mathfrak{A}}(a_0, \ldots, a_{n-1})) & = & \mathtt{f}^{\mathfrak{B}}(\phi(a_0), \ldots, \phi(a_{n-1})) & (\phi \text{ is a homomorphism}) \\ & = & \mathtt{f}^{\mathfrak{B}}(\phi(b_0), \ldots, \phi(b_{n-1})) & (\text{ assumption } a_0 \equiv b_0, \ldots) \\ & = & \phi(\mathtt{f}^{\mathfrak{A}}(b_0, \ldots, b_{n-1})) & (\phi \text{ is a homomorphism}). \end{array}$$

Therefore, we have

$$\mathtt{f}^{\mathfrak{A}}(a_0, \ldots, a_{n-1}) \equiv \mathtt{f}^{\mathfrak{A}}(b_0, \ldots, b_{n-1}).$$

□

Homomorphism theorem

Theorem (Homomorphism theorem)

Let $\phi : \mathfrak{A} \to \mathfrak{B}$ *be a surjective homomorphism. Let* $\equiv$ *be the congruence relation on* A *defined in the above lemma, that is,*

$$a \equiv b \Leftrightarrow \phi(a) = \phi(b).$$

Then there exists an isomorphism $\phi_{\equiv} : \mathfrak{A}/{\equiv} \to \mathfrak{B}$ *such that* $\phi = \phi_{\equiv} \circ \pi_{\equiv}$.

$$\begin{tikzcd}
\mathfrak{A} \arrow[r, "\phi"] \arrow[d, "\pi_{\equiv}"'] & \mathfrak{B} \\
\mathfrak{A}/{\equiv} \arrow[ur, "\phi_{\equiv}"'] &
\end{tikzcd}$$

Proof.

- We define $\phi_\equiv$ as $\phi_\equiv([a]) = \phi(a)$ for $[a] \in |\mathfrak{A}/\equiv|$,
- If $[a] = [b]$, by definition we have $a \equiv b$, and thus $\phi(a) = \phi(b)$. The converse is also true. Hence $\phi_\equiv$ is injective.
- Since ϕ is surjective, so is $\phi_\equiv$.
- Finally, we claim that $\phi_\equiv$ is a homomorphism.

$$
\begin{aligned}
\phi_\equiv(\mathtt{f}^{\mathfrak{A}/\equiv}([a_0],\ldots,[a_{n-1}])) &= \phi_\equiv([\mathtt{f}^{\mathfrak{A}}(a_0,\ldots,a_{n-1})]) && \text{(factor algebra)}\\
&= \phi(\mathtt{f}^{\mathfrak{A}}(a_0,\ldots,a_{n-1})) && \text{(definition of } \phi_\equiv\text{)}\\
&= \mathtt{f}^{\mathfrak{B}}(\phi(a_0),\ldots,\phi(a_{n-1})) && (\phi \text{ is a homomorphism})\\
&= \mathtt{f}^{\mathfrak{B}}(\phi_\equiv([a_0]),\ldots,\phi_\equiv([a_{n-1}])) && \text{(definition of } \phi_\equiv\text{).}
\end{aligned}
$$

□

The following corollary can be proved in almost the same way as the homomorphism theorem.

Corollary

Let $\phi : \mathfrak{A} \to \mathfrak{B}$ *be a homomorphism and* $\equiv$ *be a congruence relation on* $\mathfrak{A}$ *such that*

$$a \equiv b \Rightarrow \phi(a) = \phi(b).$$

Then there exists a homomorphism $\phi_{\equiv} : \mathfrak{A}/\equiv \to \mathfrak{B}$ *such that* $\phi = \phi_{\equiv} \circ \pi_{\equiv}$.

Thank you for your attention!