Logic and Foundations I, Autumn 2023

Homework No.2
Due: 2023.10.09
Name:

Problem 1
Let $\mathcal{L} = \{\mathtt{g}_1, \mathtt{g}_2, \mathtt{h}\}$. We define the set of equations E as follows.

$$E = \{\mathtt{h}(\mathtt{g}_1(x), \mathtt{g}_2(x)) = x,\ \mathtt{g}_1(\mathtt{h}(x, y)) = x,\ \mathtt{g}_2(\mathtt{h}(x, y)) = y\}$$

Let $\mathcal{K}$ be $\mathrm{Mod}(E)$, the class of models of E. Show that all finitely generated free $\mathcal{K}$-algebras are isomorphic.

Solution:

Problem 2

Boolean algebra

The theory of Boolean algebra (BA) is defined in language $\mathcal{L}_{\mathrm{B}} = \{\vee, \wedge, \neg, \mathtt{0}, \mathtt{1}\}$ with the following axioms.

(1) All the lattice axioms and the following distributive law:

$$(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z), \quad (x \wedge y) \vee z = (x \vee z) \wedge (y \vee z).$$

(2) $x \vee \mathtt{0} = x, \quad x \vee (\neg x) = \mathtt{1}, \quad x \wedge \mathtt{1} = x, \quad x \wedge (\neg x) = \mathtt{0}.$

A model of theory BA is called a Boolean algebra.

In the definition of Boolean algebra, reduce (1) to only the commutative law and distributive law, and then prove the Idempotent, absorption law, and associative law.

Solution: