Logic and Foundations I, Autumn 2023

Homework No.14
Due:
Name:

Problem 1
Let $\mathfrak{A}$ be a non-standard model of PA, and $a \in A$ be an arbitrary non-standard element. Then, in $\mathfrak{A}$, let $\mathrm{K}(\mathfrak{A}\,;a)$ denote the set of all element $b \in A$ that can be defined by the formula $\varphi(x, a)$ (does not include parameters other than a). That is, $\mathrm{K}(\mathfrak{A}\,;a)$ denote the set of b's such that $\mathfrak{A}_{\{a,b\}} \models \forall x(x = b \leftrightarrow \varphi(x, a))$. Then prove the following.

(1) By restricting functions and relations of $\mathfrak{A}$ to that of $\mathrm{K}(\mathfrak{A}\,;a)$, $\mathrm{K}(\mathfrak{A}\,;a)$ can be seen as a substructure of $\mathfrak{A}$. $\mathrm{K}(\mathfrak{A}\,;a)$ is an elementary substructure of $\mathfrak{A}$.

(2) Prove that the following set is recursive and finitely satisfiable, but it cannot be realized by $\mathrm{K}(\mathfrak{A}\,;a)$.

$$\Phi(x, a) = \{\exists v \varphi(v, a) \to \exists v < x\ \varphi(v, a) : \varphi(v, u) \text{ contains no free variables or parameters other than } u, v\}$$

<u>Solution:</u>

Problem 2

Let $\mathfrak{A} = (A, +, \cdot, 0, 1, <)$ be a non-standard model of $\mathrm{I}\Sigma_1$. Show that $\mathfrak{A}' = (A, +, 0, 1, <)$ is recursively saturated.

Solution: