Gauge Field Theory (by Hong-Jian He, Tsinghua University)
Homework, Term Papers and Final Exam

1. Homework

The homework of my Gauge Field Theory (QFT) course is classified into three categories below.

A. Regular Exercises (REx)

To tackle this type of exercises, you just have to derive some key formulas presented during the course or extend some of them to more general cases.

B. Non-Regular Exercises (NREx)

You have to derive some important results with further thinking on the basis of those presented during the course and reading some major reference books on my list. Some of them may not have been fully worked out in the existing reference books, and your independent thinking and derivations are needed.

C. Advanced Homework (AHW)

Besides those results which have been nearly worked out or have been worked out in some special cases during the course, some advanced topics are provided. Answers or results cannot be found in the existing books and references, and these topics may develop into research projects worth of journal-publication. For those students with good results, I will arrange presentations at the seminars of our Center for High Energy Physics on Wednesday or Friday.

Notes:

The **REx** is required and will amount to 30% of your Final Score; **NREx** is not required, but depending on your performance it can contribute up to 30-50% of your Final Score. The **AHW** is also not required, but depending on your outcome it can amount up to 30-70% of your Final Score. These homework will be published and updated during the course at the following Web:

http://hep.tsinghua.edu.cn/training/courses/gauge.html/exercises/

Your homeworks should be sent to my TA (Email: **gesf02@mails.thu.edu.cn**) for record, as soon as you finish for each class.

2. Term Papers

Term Papers will amount up to 30-70% of your Final Score, **in which 30% is required**.

2A. Write two Term Papers: the first one is due by 10th week (after the "Labor Day/May 1" week); the second one is due by Monday of the 16th week (just before the Final Exam).

2B. The topics of the Term Papers should be chosen based on the "Further Reading" materials I listed after each chapter. (For exceptional case, choosing a Topic outside my List may be allowed, but you have to consult with me and get my agreement in advance.) My goal is to bring you into the exciting research frontiers of Quantum Field Theories. I strongly encourage you to understand and explore some of the Nature's deep puzzles *with the knowledge you have just learnt in the class.* The contents of each Term Paper should be divided into 2 big parts: Part-1 is to summarize the key points and derivations of the paper you have read and reproduced (which cannot be simple copy/translation of its Abstract or Conclusion section, and must contain more detailed and quantitative summary); Part-2 (the Major part of your Term Paper) is to describe your further new thinking and analysis (derivations or calculations) based on the original paper.

2C. For excellent results you may have obtained from the above 1.C and 2.AB, I will arrange you to give a Presentation at the Seminars of our HEP-Center.

3. Final Exam

The Final Exam is on 17th week. The Final Exam will at most amount to 40% of your Final Score.

By the 16th week, I will announce those students who need not to attend the Final Exam due to their excellent performance in "Homeworks" and "Term Papers".

Homework by Chapters

1 Introduction

AHW : *Is path integral perfect?*

Further reading: "On Peculiarities and Pit Falls in Path Integrals", by G. 't Hooft, http://arxiv.org/hep-th/0208054

2 Chapter 1: Why Gauge Field Theories?

REx : *Causality seems violated as indicated in uncertainty principle*

The uncertainty principle indicates that:

$$\Delta p \, \Delta x \geq \hbar \tag{1}$$

This seems that causality has been violated: for small enough Δx (1) indicates that $\Delta p \geq \hbar/\Delta x$, in other words, v seems can exceed the velocity of light. How is it cured in the context of Quantum Field Theory?

NREx : *Derive the Friedmann Equation*

Using two different approaches: (1) Einstein equations and (2) Newton equations to derive the Friedmann Equation.

3 Chapter 2: Symmetries and Conservation Laws

REx : *Noether Current of QED*

Consider $U(1)$ symmetry of QED. Derive the Noether current (E.M. Current), and prove the conservation of electric charge.

NREx : *Gluinos*

Gluinos are hypothetical Majorana fermions (Superpartners of gluons). Why can they carry color-charge?

REx : *SU(2) invariant Majorana mass term*

Consider a Weyl fermion that transforms under the fundamental representation of an unbroken SU(2) group. Show that the Majorana mass term: $\frac{1}{2}m(\epsilon_{ab}\chi^{aT}\chi^b + h.c.)$ is invariant under the SU(2) symmetry. But, show that this term actually vanishes.

REx :

For $\xi \in (\frac{1}{2}, 0)$ of Lorentz group, prove that $\epsilon\xi^*$ transforms under $(0, \frac{1}{2})$ of Lorentz group, i.e. $\epsilon\xi^* \in (0, \frac{1}{2})$.

REx :

Show that $\mathcal{L}_M^m = -\frac{1}{2}m\overline{\psi}_M\psi_M$ is a Majorana mass term.

REx :

Prove that $\mathcal{L}^m = -m[(\psi^c)_L^T\hat{C}\psi_L + h.c.]$ is a Dirac mass term.

REx :

Prove that $\mathcal{L}^m = -\frac{1}{2}[\psi_L^T \hat{C} \psi_L + h.c.]$ is a Majorana mass term.

REx :

Prove that $\mathcal{L}_M^m = -\frac{1}{2}m[\overline{(\psi^c)_R}\psi_L + h.c.]$ is a Majorana mass term.

4 Chapter 3: Symmetries and their breaking

REx : *Gauge Invariant Lagrangian of Scalar Field*

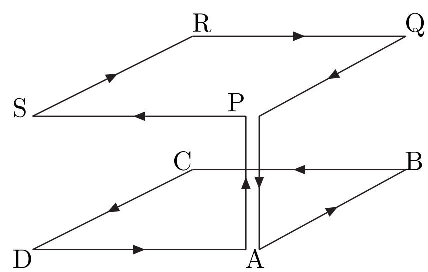
Consider a scalar field $\phi(x)$ in fundamental representation of a gauge group $\mathcal{G}$. Please construct gauge-invariant lagrangian $\mathcal{L}_{scalar}$.

REx : *Couplings in Non-Abelian Gauge Theory*

Consider the non-Abelian gauge theory $\mathcal{L} = \mathcal{L}_G + \mathcal{L}_f$. Please explicitly derive the Feynman rules for A^3, A^4 and $A - \bar{\Psi} - \Psi$ vertices in the momentum space.

NREx : *Bianchi Identity in Gauge Field Theory*

Consider the closed path as below:



Please derive a gauge field identity:

$$D_\rho F_{\mu\nu} + D_\mu F_{\nu\rho} + D_\nu F_{\rho\mu} = 0 \tag{2}$$

where:

$$D_\rho F_{\mu\nu} \equiv \partial_\rho F_{\mu\nu} - ig[A_\rho, F_{\mu\nu}] \tag{3}$$

For Abelian $U(1)$ case, equation (2) reduces to the homogeneous Maxwell Equations:

$$\partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} = 0 \tag{4}$$

Compare (2) with "Bianchi Identity" of General Relativity:

$$D_\rho R^\kappa_{\lambda\mu\nu} + D_\mu R^\kappa_{\lambda\nu\rho} + D_\nu R^\kappa_{\lambda\rho\mu} = 0 \tag{5}$$

NREx : *Correspondence between Gauge Field Theory and Differential Geometry*

The correspondence between the properties of gauge field theory and differential geometry is listed below:

Gauge Theory	Differential Geometry
Guage Transformation	Coordinate Transformation
Gauge Group	Group of all coordinate transformations
Gauge Potential $A^a_\mu T^a$	Connection Coefficient $\Gamma^\kappa_{\mu\nu}$
Field Strength $F^a_{\mu\nu} T^a$	Curvature Tensor $R^\kappa_{\lambda\mu\nu}$
Bianchi Identity: $\sum^{\rho\mu\nu}_{cyclic} D_\rho F_{\mu\nu} = 0$	Bianchi Identity: $\sum^{\rho\mu\nu}_{cyclic} D_\rho R^\kappa_{\lambda\mu\nu}$

Is there any more?

REx :
Please verify:

$$g^{\mu\nu} g_{\nu\sigma} = \delta^{\mu}_{\sigma} \tag{6}$$

NREx : *Commutator of coordinate derivatives*
We can define two sets of coordinates:

$$\begin{cases} \{x^{\mu}\}, (\mu\text{=0,1,2,3}): \text{ A General Coordinate Frame} \\ \{\xi^{m}\}, (\text{m=0,1,2,3}): \text{ The "Free Fall" Coordinate Frame} \end{cases} \tag{7}$$

and define:

$$\partial_m \equiv \frac{\partial}{\partial x^m} \qquad \partial_\mu \equiv \frac{\partial}{\partial \xi^\mu} \tag{8}$$

and:

$$e^{\mu}_{m} \equiv \frac{\partial \xi^\mu}{\partial x^m} \qquad e^{m}_{\mu} \equiv \frac{\partial x^m}{\partial \xi^\mu} \tag{9}$$

with:

$$[\partial_\mu, \partial_\nu] = 0 \tag{10}$$

One would naively expect $[\partial_m, \partial_n] = 0$, but:

$$[\partial_m, \partial_n] = [e^{\mu}_{m}\partial_\mu, e^{\nu}_{n}\partial_\nu] = (\partial_m e^{\mu}_{n} - \partial_n e^{\mu}_{m}) e^{p}_{\mu} \partial_p \tag{11}$$

which shows that the notation ∂_m is slightly misleading. Something must be different. Please figure it out.

NREx : *Construct Gauge Invariant Action for Dirac Spinor*
Extend our construction of generally covariant action $S(\phi$ and construct a generally covariant action $S(\psi, \bar{\psi})$ for Dirac spinor $(\psi, \bar{\psi})$.

AHW :
Can one build up a unified EW model with $SU(3)$ broken down to $SU(2)_w \times U(1)_y$? Why? Derive the mass formula for the 4 massive gauge boson.

5 Chapter 4: Path Integral Quantization - Gauge Fields

REx : *Deriving Feynman Rules:*
Please derive the full Feynman rules of Abel Higgs Model in $\mathcal{R}_\xi$ gauge.

NREx : *BRST Transformation* Please extend the following conclusion:

$$F^a = f^{ab}_{\mu} A^{b\mu} \tag{12}$$

that is to say, F^a is at most linear in $A^{b\mu}$, to the case of $\mathcal{R}_\xi$ gauge where F^a contains Goldston boson π^a.

REx : *BRST Symmetry Charge*
Proof that the square of the charge of BRST symmetry transformation vanishes:

$$Q^2_{BRST} = 0 \tag{13}$$

REx : *BRST symmetry*

Prove that BRST transformations are nilpotent:

$$\hat{\delta}^2 = 0 \qquad Or \qquad \hat{s}^2 = 0 \tag{14}$$

REx : **WT Identity - a**

Derive the following relations:

$$\frac{\delta W}{\delta J^\mu} = A_\mu \tag{15}$$

$$\frac{\delta W}{\delta \bar{I}} = \psi \tag{16}$$

$$\frac{\delta W}{\delta I} = -\bar{\psi} \tag{17}$$

$$\frac{\delta \Gamma}{\delta A^\mu} = -J_\mu \tag{18}$$

$$\frac{\delta \Gamma}{\delta \psi} = -\bar{I} \tag{19}$$

$$\frac{\delta \Gamma}{\delta \bar{\psi}} = I \tag{20}$$

and Ward-Takahashi Identity:

$$\left[ie\left(\bar{I}\frac{\delta}{\delta \bar{I}} - I\frac{\delta}{\delta I}\right) + \frac{1}{\xi}\partial^2\partial^\mu \frac{\delta}{\delta J^\mu}\right] W(J, I, \bar{I}) = \partial_\mu J^\mu \tag{21}$$

REx : *WT Identity - b*

Derive the following Ward-Takahashi Identities:

$$\partial_\mu \frac{\delta \Gamma}{\delta A_\mu} + e\left(\bar{\psi}\frac{\delta \Gamma}{\delta \bar{\psi}} - \psi\frac{\delta \Gamma}{\delta \psi}\right) + \frac{i}{\xi}\partial^2\partial^\mu A_\mu = 0 \tag{22}$$

and:

$$q^\mu \Gamma_\mu(p, q, p+q) = S_F^{-1}(p+q) - S_F^{-1}(p) \tag{23}$$

6 Chapter 5: Renormalization of Gauge Theories

REx : *Derive Superficial Divergence Degree*

$$D = 4 - E_A - \frac{3}{2}E_F \tag{24}$$

NREx : *Proof that:*

$$\{\mathcal{G}_0 + \mathcal{G}_1\} + \mathcal{G}_1^2 = 0 \tag{25}$$

with:

$$\mathcal{G}_0 \equiv \frac{\delta \tilde{\Gamma}^{(0)}}{\delta K_i}\frac{\delta}{\delta \phi_i} + \frac{\delta \tilde{\Gamma}^{(0)}}{\delta L^a}\frac{\delta}{\delta c^a} \tag{26}$$

$$\mathcal{G}_1 \equiv \frac{\delta \tilde{\Gamma}^{(0)}}{\delta \phi_i}\frac{\delta}{\delta K_i} + \frac{\delta \tilde{\Gamma}^{(0)}}{\delta c^a}\frac{\delta}{\delta L^a} \tag{27}$$

NREx : *Please derive*

$$\Gamma^{(n)}_{div} = \Bigg[\left(\frac{1}{2}\alpha_{div}+\beta_{div}\right)\left(\phi_i\frac{\delta}{\delta\phi_i}+L^a\frac{\delta}{\delta L^a}\right) -\frac{1}{2}\left(\gamma_{div}+\beta_{div}\right)\left(c^a\frac{\delta}{\delta c^a}+\bar{c}^a\frac{\delta}{\delta\bar{c}^a}+K_i\frac{\delta}{\delta K_i}\right)-\frac{1}{2}\alpha_{div}g\frac{\partial}{\partial g}\Bigg]\Gamma^{(0)}(\phi,c,\bar{c},K,L,g) \tag{28}$$

using the following two equations:

$$\Gamma^{(n)}_{div}=\alpha_{div}S[\phi]-\beta_{div}(K_i+\partial^a_i\bar{c}^a)D^b_ic^b+\gamma_{div}L^a\frac{\delta\Gamma^{(0)}}{\delta L^a}+\beta_{div}\frac{\delta\Gamma^{(0)}}{\delta\phi_i}\phi_i+\gamma_{div}\frac{\delta\Gamma^{(0)}}{\delta c^a}c^a \tag{29}$$

and:

$$S[\phi,g]=\frac{1}{2}\phi_i\frac{\delta S}{\delta\phi_i}-\frac{1}{2}g\frac{\delta S}{\delta g} \tag{30}$$

REx : *Derive the running equation of coupling constant:*

$$g(\mu)=\frac{g(\mu_0)}{\left[1-(n-1)bg^{n-1}(\mu_0)\ln\frac{\mu}{\mu_0}\right]^{\frac{1}{n-1}}} \tag{31}$$

REx : *Derive the Callan-Symanzik Equation:*

$$\left[m\frac{\partial}{\partial m}+\beta\frac{\partial}{\partial g}-n\gamma\right]\Gamma^{(n)}(p_i,g,m)=-i\alpha m^2\Gamma^{(n)}_{\phi^2}(0,p_i,g,m) \tag{32}$$

with:

$$\beta \equiv 2m^2\frac{\partial g}{\partial m_0^2}\left(\frac{\partial m^2}{\partial m_0^2}\right)^{-1} \tag{33}$$

$$\gamma \equiv m^2\frac{\partial\ln Z_\phi}{\partial m_0^2}\left(\frac{\partial m^2}{\partial m_0^2}\right)^{-1} \tag{34}$$

$$\alpha \equiv 2Z_{\phi^2}\left(\frac{\partial m^2}{\partial m_0^2}\right)^{-1} \tag{35}$$

REx : *Derive the following formulas for γ and γ_m in the Minimal Subtraction Scheme:*

$$\gamma=\frac{\partial\ln Z_\phi^{\frac{1}{2}}}{\partial\ln\mu}=gc_1' \qquad c_1'\equiv\frac{dc_1}{dg} \tag{36}$$

$$\gamma_m=\frac{\partial\ln m}{\partial\ln\mu}=gb_1' \qquad b_1'\equiv\frac{db_1}{dg} \tag{37}$$