Tutorial

Given are a set of n values $x_1, x_2, \ldots, x_n$ with errors $\sigma_i, i = 1, \ldots, n$.
Determine the weights $w_i, i = 1, \ldots, n$ of the linear combination $S = \sum_i w_i x_i$ which minimizes $V = \sum_i w_i^2 \sigma_i^2$ under the constraint $\sum_i w_i = 1$.

a) For $n = 2$ determine the optimum by substituting $w_2 = 1 - w_1$.

b) For $n = 2$ determine the optimum using the Langrange multiplier method.

c) Generalize b) to arbitrary values of n.

An accelerator produces a beam consisting to 90% of pions and 10% of muons. A muon-nucleon scattering experiment uses a trigger which can discriminate between the two particle types, with a probability to trigger on a muon $p(T|\mu) = 0.95$ and a probability to trigger on background $p(T|\pi) = 0.02$. How large is the fraction of true muons in the trigger.

The host of a game show tell a candidate that there is a treasure behind one of three doors, while the room is empty behind the others. The door with the treasure is of course not known. Now the candidate is allowed to select one door - but not yet to open it. The candidate selects one of the doors. Then the host opens one of the other two doors, behind which there is nothing. The candidate now has the choice to either change his selection or to stick to the original choice. What should he do in order to maximize the chance to obtain the treasure?

Label without loss of generality the door selected by the candidate by "1", the door opened by the host by "3" and the alternative door for the candidate by "2". The prior probabilities for the treasure behind door "i" are $p(T_i) = 1/3$.
The probability that the host opens door 3 if the treasure is behind door 1 is $p(O_3|T_1) = 1/2$, that he opens door 3 if the treasure is behind door 2 is $p(O_3|T_2) = 1$, and the he opens door 3 if the treasure is behind door 3 is $p(O_3|T_3) = 0$.

For the probability to find the treasure behind door 1 Bayes' theorem yields:

$$p(T_1|O_3) = \frac{p(O_3|T_1)p(T_1)}{p(O_3|T_1)p(T_1) + p(O_3|T_2)p(T_2) + p(O_3|T_3)p(T_3)}$$
$$= \frac{p(O_3|T_1)}{p(O_3|T_1) + p(O_3|T_2) + p(O_3|T_3)} = \frac{1}{3}$$

For the probability to find the treasure behind door 2 Bayes' theorem yields:

$$p(T_2|O_3) = \frac{p(O_3|T_2)p(T_2)}{p(O_3|T_1)p(T_1) + p(O_3|T_2)p(T_2) + p(O_3|T_3)p(T_3)}$$
$$= \frac{p(O_3|T_2)}{p(O_3|T_1) + p(O_3|T_2) + p(O_3|T_3)} = \frac{2}{3}$$

Central limit theorem

Histogram the sum of n random numbers $x_i, i = 1, \ldots, n$, with x_i distributed uniformly over the range $-a_i < x_i < a_i$. Consider the following cases:

a) $n = 2$ and $a_i = 1$

b) $n = 10$ and $a_i = 1$

c) $n = 10$ and $a_i = \sqrt{i}$

For case a) also calculate and compare to the analytical expectation. For cases b) and c) also overlay the expectation derived from the central limit theorem.

Test the central limit theorem for random numbers y generated by $y = \mathtt{tan}(x\pi/2)$, where x is uniformly distributed in $[-1, 1]$.

a) plot the distribution of y

b) plot the convolution of n random numbers $y_i, i = 1, \ldots, n$, for $n = 2, 4, 8, \ldots$

Convolutions

Generate pairs (x, y) of gaussian distributed random variables, both with a mean value $\mu = 0$ and $\sigma = 1$, and for each pair calculate $z = \sqrt{x^2 + y^2}$.

a) Histogram the distribution of z.

b) Write down the analytical expression for the integral solved by the algorithm.

c) solve the integral analytically and compare to the simulation

➜ *solution to b) and c)*

$$\int dx \int dy \frac{1}{\sqrt{2\pi}} e^{-x^2/2} e^{-y^2/2} \, \delta(z - \sqrt{x^2 + y^2})$$

$$= \frac{1}{2\pi} \int dx \int dy e^{-(x^2+y^2)/2} \, \delta(z - \sqrt{x^2 + y^2})$$

$$= \frac{1}{2\pi} \int_0^{2\pi} d\phi \int_0^{\infty} dr \; r \; e^{-r^2/2} \, \delta(z - r)$$

$$= z \; e^{-z^2/2}$$

Random numbers

Use the Hit-or-Miss method to generate gaussian random numbers in the range $-5 < x < 5$.

```
double f = 0.;
double y = 1;
double x = 0.;
while(y>f) {
  x = rndm(-5.,5.);
  f = exp(-x*x/2.);
  y = rndm(0.,1);
}
return x;
```

Determine a function $y = h(x)$ which maps uniformly distributed random numbers from $0 < x < 1$ to random numbers y with the singular PDF $\rho(y) = 1/2\sqrt{y}$ in the range $0 < y < 1$. Write a ROOT macro which fills a histogram with 10^6 random numbers generated according to $\rho(y)$.

$$h^{-1}(y) = \int_0^y dx \, \frac{1}{2\sqrt{x}} = \sqrt{y} \quad \rightarrow \quad h(y) = y^2$$

```
TH1D *h1 = new TH1D("h1","h1",100.,0.,1.);
for(int i=0; i<100000; ++i) {
  h1->Fill(pow(rndm(0.,1.),2));}
}
h1->Draw()
```

Write a function `int poisson(double mu)` which for a given mean value μ generates poissonian distributed random numbers. The probability to observe the integer $n \geq 0$ is given by:

$$p_n = e^{-\mu} \frac{\mu^n}{n!}$$

```
double x  = rndm(0.,1.);
double pn = exp(-mu);
double S  = pn;
int    n  = 0;
while(S<x) {
  pn *= mu/n;
  S  += pn;
}
return n;
```

Do a Monte Carlo integration of a gaussian PDF over the range $-3 < x < +3$.

a) using 10^6 uniform random numbers over the range $-3 < x < +3$

b) using 10^6 gaussian random numbers over the range $-\infty < x < +\infty$

c) how many random numbers would one need in case a) to obtain the same precision as in case b)

Importance sampling

Calculate the integral $\int_0^1 dx \, \frac{e^{-x}}{\sqrt{x}}$.

a) Try to do use uniform random numbers.

b) Use importance sampling.

➜ *solution for a)*

```
int     N  = 100000;
double S1 = 0.;
double S2 = 0.;
for(int i=0; i<N; ++i) {
   x    = rndm(0.,1.);

   f    = exp(-x)/sqrt(x);
   S1 += f;
   S2 += f*f;
}
double  I = S1/N;
double dI = sqrt((S2/N)-I*I)/N);
```

➜ *solution for b)*

```
int     N  = 100000;
double S1 = 0.;
double S2 = 0.;
for(int i=0; i<N; ++i) {
   x   = rndm(0.,1.);
   x  *= x
   f   = 2.*exp(-x);
   S1 += f;
   S2 += f*f;
}
double  I = S1/N;
double dI = sqrt((S2/N)-I*I)/N);
```