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A STATISTICAL MODEL FOR PHOTOMULTIPLIER SINGLE-ELECTRON STATISTICS

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A statistical model is proposed to describe secondary electron emission in photomultipliers, based on the Polya or "negative binomial" distribution, which contains the exponential and Poisson distributions as special cases. Computations of the single-electron spectra have been carried out for a variety of stage-gains. Limited comparisons with available experimental data confirm that the *effective* statistical distribution of secondary emission in photomultipliers is broader than Poissonian and an interpretation in terms of non-uniformity of the dynodes is discussed.

1. Introduction

In discussing the statistics of multiplication in photomultipliers, it has frequently been assumed that an appropriate statistical frequency function for describing secondary electron emission at individual dynodes is the Poisson distribution. In particular, most of the theoretical discussions[1–7]) of the problem have made this assumption.

In one of the more recent of these, Lombard and Martin[5]) computed the statistical distributions at the anode of the pulses arising when a single electron enters the dynode chain, for four different values of stage gain, assumed equal for each dynode. They remarked that the calculated distributions were not consistent with their experimentally observed single-electron distributions. In brief, the calculated curves are definitely peaked, whereas observed distributions were usually exponential or at least monotonically decreasing. Lombard and Martin concluded that the Poisson distribution is not an adequate model to apply to secondary emission statistics. This type of quasi-exponential experimental distribution has been reported by a number of other observers[8–14]).

On the other hand there is a substantial body of published data in which the single-electron spectrum is clearly peaked[10,12,15–28]).

It is clear from these latter references, firstly: that care should be taken to distinguish single-electron pulses produced by light incident on the cathode from pulses arising from tube background. While, for selected tubes, the single-electron spectra from these two sources may be almost identical[12]), in others, the two spectra are quite different[15,17,19]).

Secondly: The work of Delaney and Walton[10]) and of Koechlin[15]) has shown that light penetrating the photocathode and ejecting electrons from the first and subsequent dynodes may produce sufficient small pulses that the single-electron spectrum is very much distorted. The former also show that the spectrum may depend on where the photocathode is illuminated, see also[28]).

With some exceptions, then, spectra from single electrons, released from the photocathode by light, show a well defined peak, and it appears that the quasi-exponential spectra that have been obtained are not necessarily typical single-electron spectra.

This is not to say, however that the Poisson distribution is necessarily the correct statistical model to describe dynode statistics in photomultipliers. To begin with, the secondary emission process itself is not necessarily Poissonian, see e.g. Breitenberger[29]) and in the second place, non-uniformity of the dynodes can easily render the *effective* statistical distribution of secondary electrons non-Poissonian[7]). It is the purpose of the present paper to examine the possibilities of a non-Poissonian statistical model for photomultiplier dynode statistics. The model covers either of the two preceding possibilities.

2. Theory

We consider a single-electron striking the first dynode of a photomultiplier of K identical stages. Although it is relatively simple to write down explicitly the moments of the distribution at the output[3,4]) the corresponding frequency function cannot in general be written down in closed form. Let $f(x)$ be the frequency function for secondary emission of exactly x secondaries for each stage; and let

$$G(s) = \sum_{\text{all } x} s^x f(x)$$

be the corresponding generating function, where s is an auxiliary variable. The generating function, $G_K(s)$, for the output of the K^{th} stage is then given by:

$$\begin{aligned} G_K(s) &= G[G\{\ldots. \quad G(s)\}]\ldots. \qquad K \text{ stages} \\ &= G[G_{K-1}(s)]. \end{aligned}$$

Replacing $\ln s$ by is yields the characteristic function $\chi_K(s)$, which is the Fourier transform of the frequency function of the output from the K^{th} stage. $\chi_K(s)$ is usually too complicated for the Fourier transform to be inverted.

In at least one case however, which approximates more or less closely to the quasi-exponential frequency functions, the solution can be found explicitly without the use of transforms, viz. if $f(x)$ is the Furry distribution: $f(x)=a^{-1}(1-a^{-1})^x$, with mean $= a-1$ and generating function $\{a-s(a-1)\}^{-1}$. Application of the foregoing theorem on generating functions in cascade shows that any number of Furry distributions in cascade generates another Furry distribution, with a mean after K stages of $(a-1)^K$. If $(a-1)^K \gg 1$, which is the case in a photomultiplier, the distribution approaches a simple exponential with mean A:

$$F(x) = A^{-1}\,\mathrm{e}^{-x/A}.$$

Since at least in some cases, single-electron distributions are, in fact, similar to exponentials, it is evident that at least in these cases the Furry distribution is a plausible model for dynode secondary emission statistics. A similar suggestion has recently been made by Baldwin and Friedman[14]). However, a more generally valid model must still be sought for.

To take account of the more general case, we suggest the use of the Polya distribution:

$$P(x) = \frac{\mu^x}{x!}(1+b\mu)^{-x-1/b}\prod_{i=1}^{x-1}(1+ib), \tag{1}$$

with generating function:

$$G(s) = \{1+b\mu(1-s)\}^{-1/b}, \tag{2}$$

where b is a parameter and μ the mean of the distribution, see e.g. Feller[30]). Known also as the "negative binomial distribution" and also as a "compound Poisson" distribution, it contains both the Furry ($b=1$) and Poisson ($b=0$) distributions as special cases. It has been used in the interpretation of cosmic ray shower fluctuations[32,31]) and fluctuations in gas multiplication in proportional counters[33]). While these circumstances originally suggested its use in the analogous problem of cascades in a photomultiplier, it is not difficult to see that the model might also be of value in describing the fluctuations in the secondary emission process itself. Furthermore its so-called "compound Poisson" character lends itself to a simple physical interpretation in the context of photomultiplier statistics, permitting the inclusion of non-uniform dynodes in the theoretical treatment in a relatively simple way, as follows: Let us suppose, for the sake of argument, that the secondary emission process *is* truly Poissonian, but that the effective secondary emission ratio varies from place to place on each dynode, either because the average number of electrons actually ejected is variable, or the emitted electrons do not all have the same probability of reaching the following dynode.*

Let each electron falling on a dynode produce secondaries, Poisson-distributed with mean m. Further, let the values of m be samples from a continuous distribution of mean values represented by the frequency function $g(m)$. Since the generating function for a Poisson distribution having a mean m is given by $\exp\{m(s-1)\}$ the generating function, $G(s)$, for the compound Poisson distribution is given by:

$$G(s) = \int_0^\infty g(m)\exp\{m(s-1)\}\,\mathrm{d}m.$$

Now, the moment generating function $M_m(s)$ for m is given by:

$$M_m(s) = \int_0^\infty g(m)\exp(ms)\,\mathrm{d}m.$$

Hence

$$G(s) = M_m(s-1). \tag{3}$$

This neat result is unique with compound Poisson distributions. Eq. (3) leads to the result that the mean of the compound Poisson distribution is the same as the mean of the distribution, $g(m)$ and that the relative variance V, [variance/(mean)^2] of the compound Poisson distribution is given by:

$$V = \mu^{-1} + V_m$$

where V_m is the relative variance of the frequency function of the means, $g(m)$. Since μ^{-1} is the relative variance of the simple Poisson distribution, this reveals the not unexpected result that the width of the final distribution, as measured by its relative variance, is increased by exactly the spread of the means themselves. Applying eq. (3) to the particular case of the Polya distribution

$$G(s) = \{1+b\mu(1-s)\}^{-1/b}$$

and

$$M_m(s) = (1-b\mu s)^{-1/b},$$

which is the moment generating function of the Laplace distribution, viz.

$$g(m) = \frac{(\mu b)^{-\alpha}}{\Gamma(\alpha)}\,m^{\alpha-1}\exp(-m/\mu b),$$

where $\alpha = b^{-1}$.

* It should be noted that loss of electrons between stages does not of itself result in non-Poissonian effective dynode statistics. If the loss is a constant fraction, then a Poisson distribution remains Poissonian though with a reduced mean.

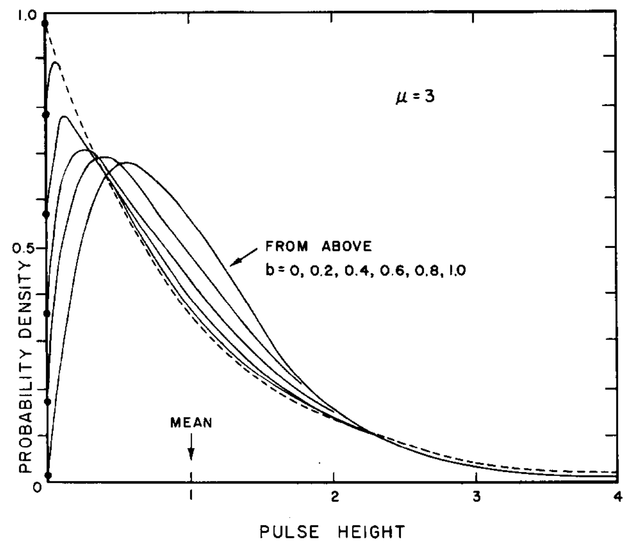

Fig. 1. Computed single-electron distributions at stage-gain 3.0, for a range of values of parameter b.

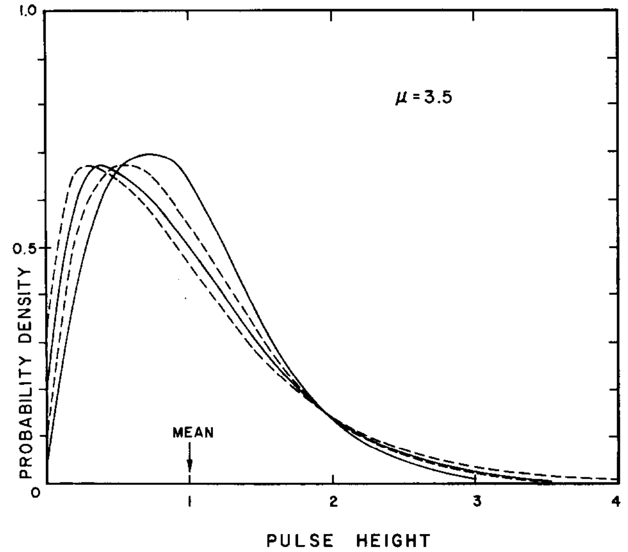

Fig. 2. Computed single-electron distributions at stage-gain 3.5. Parameter b: 0, 0.1, 0.2, 0.3.

The mean of this distribution is μ, the variance $b\mu^2$ and the relative variance b. Except for $b = 1$, this is a peaked distribution which gets narrower as the parameter b gets smaller. In the limit of $b = 0$ it becomes a delta-function at $m = \mu$, which, in the present context, simply means that there is no variation in the secondary emission ratio across the surface of a dynode. In the other limit, $b = 1$, the distribution $g(m)$ becomes exponential and the secondary emission distribution is the Furry distribution. This is the special case discussed by Baldwin and Friedman[14]).

In the present context, then, the Polya statistical model describes secondary emission from a dynode of finite area for which the average number of secondaries per incident electron for the dynode *as a whole* is μ. For any one point on the dynode the probability distribution of the secondaries is Poissonian, but the *local* value of the mean number of secondaries varies from place to place in a manner described by the Laplace distribution. The relative variance of the distribution of all secondary electrons reaching the following dynode is then greater than it would be if the statistical spread were due only to the random (assumed Poisson) nature of the secondary emission. In particular, for the Polya distribution, the relative variance is given by: $V = \mu^{-1} + b$.

It should be noted that in all of the foregoing theoretical expressions, "blanks" are included, i.e. cascades that break for lack of any secondary electrons. In practice such blanks are inherently unobservable.

This particular non-Poisson model and the following calculations based on it, could equally well apply to the secondary emission process itself, observed under ideal conditions. In this case however its interpretation as a "compound Poisson" distribution is not obvious.

3. Calculations

Using the Polya distribution as a model for dynode statistics, the single-electron spectrum has been computed numerically by a generalization of the method used by Lombard and Martin[5]). Equal stage gains are assumed.

Let $P_K(x)$ and $G_K(s)$ represent the frequency and generating functions respectively for the distribution of the total number of electrons, x, after stage K. Then,

$$G_K(s) = G_1\{G_{K-1}(s)\} = \sum_{\text{all } x} P_K(x)s^x \qquad (K \geqq 1).$$

This expresses $G_K(s)$ as a Maclaurin series in $P_K(s)$ since

$$P_K(x) = \frac{1}{x!}\,G_K^{(x)}(s) = \frac{1}{x!}\,[G_1\{G_{K-1}(s)\}]^{(x)}, \qquad (4)$$

evaluated at $s = 0$, where $G_K{}^{(x)}(s)$ is the x^{th} derivative of $G_K(s)$, ref. [32,30]). Substituting (2) for $G(s)$ in (4) and applying Leibniz' rule for the differentiation of a product, viz.

$$(fg)^{(x)} = \sum_{i=0}^{x} \binom{x}{i} f^{(i)} g^{(x-i)},$$

and after rearrangement of terms within the sums, the following iteration formulae for $P_K(x)$ are obtained:

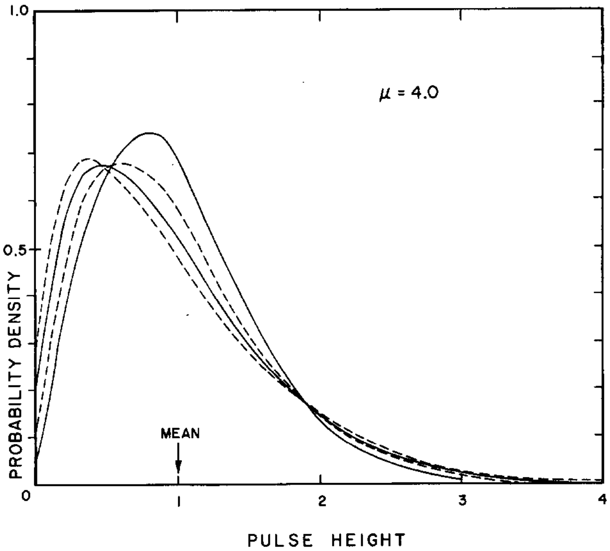

Fig. 3. Computed single-electron distributions at stage-gain 4.0. Parameter b: 0, 0.1, 0.2, 0.3.

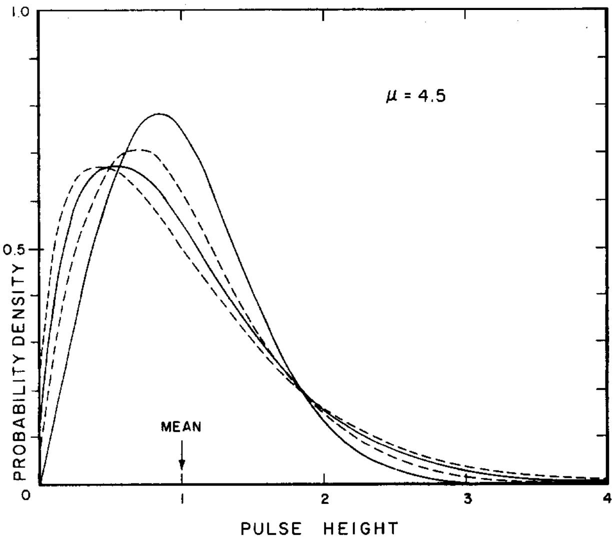

Fig. 4. Computed single-electron distributions at stage-gain 4.5. Parameter b: 0, 0.1, 0.2, 0.3.

$$xP_K(x)\,\{1+b\mu[1-P_{K-1}(0)]\}$$
$$=\mu\sum_{i=0}^{x-1}P_K(i)P_{K-1}(x-i)\,\{x+i(b-1)\}\text{ for }K\geqq 1;\ x\geqq 1 \tag{5}$$

$$P_K(0)=\{1+b\mu[1-P_{K-1}(0)]\}^{-1/b}. \tag{6}$$

If $b=0$, i.e., at the Poisson limit of the Polya distribution, expressions (5) and (6) reduce to expressions (7) and (8) of Lombard and Martin[5]). It is easily checked that if $K=1$, the above expressions yield the Polya distribution itself. The initial condition, viz. one electron incident on the first dynode, is expressed by the relation:

$$P_0(x)=1\text{ for }x=1\text{ and zero otherwise.}$$

For the same distribution, M_K^l, the l^{th} moment about the origin after stage K, is given by:

$$M_K^l=\sum_{x=0}^{\infty}x^lP_K(x)$$
$$=\mu M_{K-1}^l+\mu\sum_{j=1}^{l-1}\left\{b\binom{l-1}{j}M_K^{l-j}M_{K-1}^j+\right.$$
$$\left.+\binom{l-1}{j}M_{K-1}^{l-j}M_K^j\right\}.$$

The relative variance, V, after stage K is:

$$V=(b+\mu^{-1})\,(1+\mu^{-1}+\mu^{-2}+\ldots\mu^{-K+1})$$

which, in the limit of large K, becomes

$$V=(b\mu+1)/(\mu-1).$$

Expressions (5) and (6) were evaluated on the IBM 7090 at the University of Toronto for a range of stage gain (parameter μ in expression 1) and parameter b in the same expression. For each stage the iteration was terminated when

$$\sum_{x=0}^{x}P_K(x)>0.999,\ (0.9999\text{ for }K=1),$$

$$\text{or when }P_K(x)<10^{-10}.$$

As is well known, the first stage of multiplication has a dominant influence on the shape of the final output spectrum and subsequent stages have a rapidly decreasing importance[7,18]). In the present calculations the iteration was usually terminated at the fifth stage,

TABLE 1
Fraction k of cascades that fail to propagate.

Polya parameter, b	Mean stage gain, μ 3.0	3.5	4.0	4.5	5.0	6.0
0	0.060	0.034	0.020	0.012	0.007	0.003
0.1	–	0.058	0.039	0.026	0.018	0.009
0.2	0.12	0.084	0.060*	0.045	0.034	0.020
0.3	–	0.11	0.084	0.066	0.052	–
0.4	0.18	–	0.11	–	–	–
0.5	0.21	–	0.13	–	–	–
0.6	0.24	–	–	–	–	–
0.8	0.29	–	–	–	–	–
1.00	0.33	–	–	–	–	–

Additional value, μ = 2, b = 0.5; k = 0.38.
* Interpolated value.

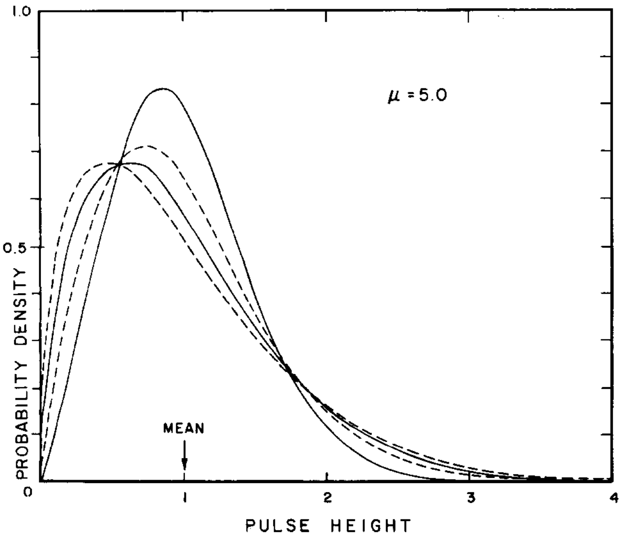

Fig. 5. Computed single-electron distributions at stage-gain 5.0. Parameter b: 0, 0.1, 0.2, 0.3.

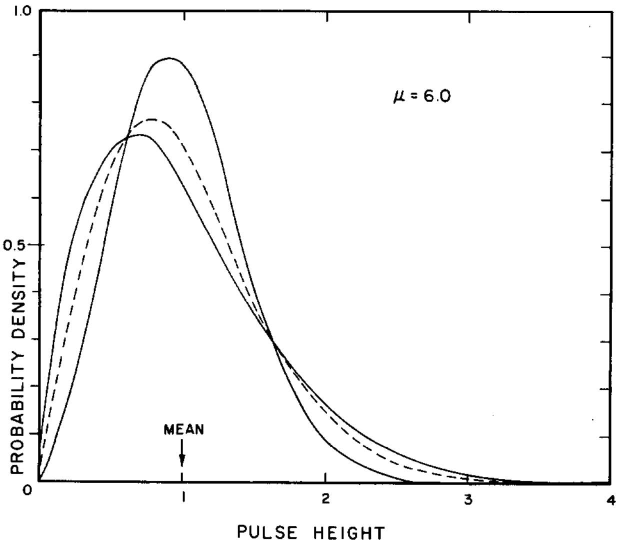

Fig. 6. Computed single-electron distributions at stage-gain 6.0. Parameter b: 0, 0.1, 0.2.

and for $\mu \geq 4.5$, three stages of iteration were sufficient. Since the programme printed the spectrum after each iteration, the rapid stabilisation of the spectrum was readily confirmed.

The results of the calculations are shown in fig. 1–6 and in table 1. In each case, to facilitate comparisons with one another and with experiment, the curves have been plotted with the mean at unity on the abcissa and with the same normalization. *Neither* this mean *nor* the normalization includes blanks, i.e. cascades that fail to propagate. Table 1 shows the fraction, k, of such cascades, i.e. the value of $P_K(0)$, for most of the curves computed, including some not shown in the figures. Such cascades are unobservable although the computer programme, of course, takes account of them. The mean μ' and relative variance V' of the *observable* distribution are related to the *computed* mean μ and relative variance V by the expressions:

$$\mu' = \mu/(1-k);$$

$$V' = V(1-k)-k = [(b\mu+1)(1-k)/(\mu-1)]-k.$$

Note also that all curves have a non-zero intercept on the ordinate. This is, of course, not the probability of observing zero electrons at the output but, for all practical purposes, is very nearly the probability of observing just one electron.

Fig. 1 shows a family of curves for $\mu = 3$ and a complete range of values of b. Other curves, for a range of values of μ, but restricted values of b, are shown in fig. 2–6.

4. Discussion

Examination of the data displayed in the figures shows that the Polya model, proposed here, is capable of representing a wide range of shapes for the single-electron distribution. It seems particularly attractive in that it contains as special extreme cases both the Poisson and exponential (Furry) distributions, both of which have been invoked in the literature to describe the performance of photomultipliers. Fig. 1 clearly shows how the shape of the computed single-electron distribution changes continuously from a distinctively peaked distribution in the Poisson ($b = 0$) limit to an exponential in the $b = 1$ limit. It is clear that already for $b \gtrsim 0.5$ the distribution is approaching an exponential: the peak has moved to small pulse-heights and the distribution has a long quasi-exponential tail. (This is very evident on a semi-logarithmic plot). Rather careful experimental measurements would be required to demonstrate the existence of this peak for values of b in the upper half of its range.

While the distributions are clearly more sensitive to changes in b when that parameter is small, it is instructive to consider what this means in terms of the physical situation described by the present model. As shown in section 2, b may be interpreted as a measure of the non-uniformity of the effective secondary emission coefficient, m, across the surface of a dynode; specifically, b is the relative variance of the distribution of m. Thus, a value for b of 0.2 means a relative standard deviation in m of about 45%. Conversely a 10% relative standard deviation corresponds to a value for b of only

0.01, which would be barely distinguishable from a Poisson distribution. Thus, irregularities as big as, say 10–15% in the dynode emission are not likely to result in any significant difference in overall performance. The present work therefore confirms the conclusion of Wright[7]) obtained by a different argument.

Of the differing varieties of photomultipliers made, the elaborately focussed types (e.g. 56AVP) are the ones least likely to be affected by dynode inhomogeneities, since the object of the electron optical system is to focus electrons as nearly as possible on the same (small) region of the dynodes, particularly the first[19,26–28]). Bertolaccini and Cova[17]) and Evrard and Gazier[27]) give data for the 56AVP from which they conclude that, while the single-electron spectrum is close to that expected from Poisson statistics, it is nevertheless significantly broader. Data from[17]) and[27]) are shown in fig. 7 where they are compared with the present calculations. The two lines show the calculated *observable* overall relative variance as a function of the stage-gain, for a multiplier with equal stage-gains, for $b = 0$ and $b = 0.1$. The circles have been taken from fig. 6 of[17]) and the squares from fig. 4 of[27]), [fig. 16 of[28])]. In the latter case the stage-gain shown is that of the first stage. The remaining stages were operated at a standard gain, and the plotted relative variance, V, has therefore been adjusted to the value to be expected if all stages had the same gain by means of the relation[28]):

$$V = V_1 + \frac{v}{\mu_1}\left(\frac{\mu}{\mu - 1}\right)$$

where V_1, μ_1 are the relative variance and gain of the first stage and v and μ the variance and gain common to all other stages. For the purposes of this correction, Poisson statistics were assumed for all stages except the first, i.e. $v = \mu^{-1}$. While this assumption may appear to prejudge the issue, it should be realized that it applies only to the *correction* which is not sensitive to it. *No* assumption is made about the statistics of the first stage which mainly determines the outcome. The correction reduces the relative variance in all cases; the original (unadjusted) values are shown by triangles.

Fig. 7 shows also that both sets of experimental data agree in suggesting a distribution broader than Poissonian, although still similar to the latter. From this figure, the value of b is estimated to be 0.05 ± 0.02, averaged over the whole range of stage-gains. There is no way of telling, from the present data, whether this is due to non-uniformity of the dynodes or to non-Poissonian statistics for the secondary emission process itself.

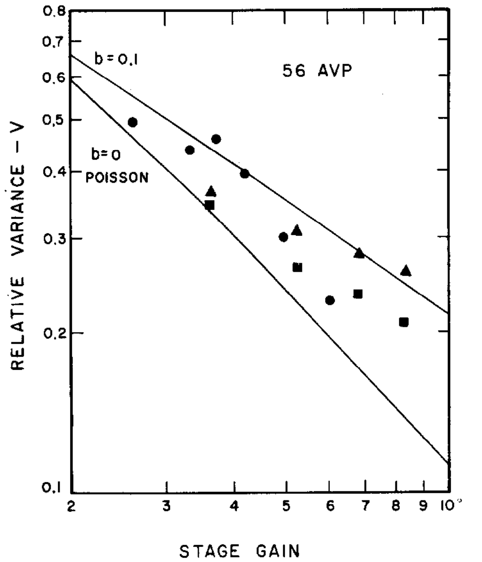

Fig. 7. Comparison of computed curves of relative variance vs stage-gain for $b = 0$ and $b = 0.1$ with experimental data of Bertolaccini and Cova[17]) (circles) and Evrard and Gazier[27]) (triangles) for 56AVP photomultipliers. Squares denote the data of Evrard and Gazier after adjustment.

So far as a detailed check of the *shape* of the single-electron spectrum is concerned, it is unfortunate that there is almost no published experimental data with which the present theory may be checked: either because the stage-gain is not accurately known or because the multiplier was not operated with equal stage-gains.*

In fig. 8, experimental data from Bertolaccini and Cova[17]) are compared with the theoretical model, using the stage-gains (2.6 and 6), given in[17]) and evaluated for values of b of 0.0 and 0.05. It is seen that for both stage-gains, the experimental data are fitted better, though not perfectly, by the curve for $b = 0.05$; which is in agreement with the previous deduction from the relative variances (fig. 7).

In fig. 9 is shown the experimental data of Delaney and Walton[10]) for an EMI 9514A of Venetian-blind structure at a stage-gain of 3.2, together with theoretical curves for the same gain and two values of b, viz. 0.0 and 0.2. Evidently the theoretical curve assuming Poisson statistics ($b = 0$) is a very poor fit. Indeed, a value for b closer to 0.2 appears to be required. The geometrical structure of this type of photomultiplier ensures that electrons from different parts of a dynode are collected with different efficiencies and also that electrons from the previous stage fall upon the whole

* While corrections for the latter fact can be readily applied to relative variances, (as was done above) the single-electron spectrum itself cannot be so corrected.

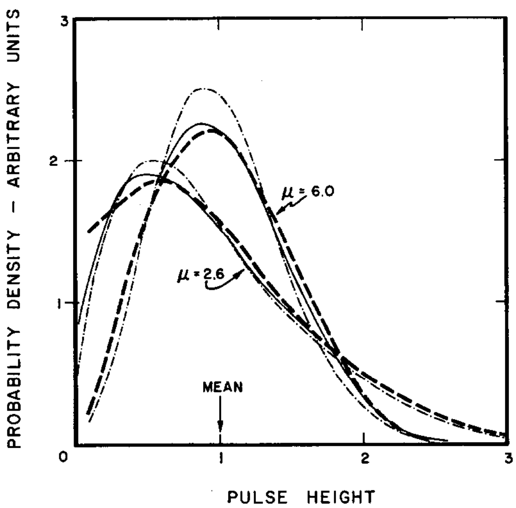

Fig. 8. Comparisons of experimental single-electron spectra for a 56AVP photomultiplier (dashed line) at two values of stage-gain, with computed curves for $b = 0.05$ (solid line) and $b = 0$ (dash-dot line).

area of the dynode. Thus it may be expected that, if dynode inhomogeneities do, in fact, play a part, then they should be more in evidence with the Venetian-blind dynode structure than with a well-focussed type. The much larger value of b for the former tube suggests, that, in fact, there is a substantial degree of dynode inhomogeneity.

There does not appear to be sufficient evidence to say whether dynode inhomogeneities alone are sufficient to account for the effectively non-Poisson dynode statistics in photomultipliers, or whether the secondary emission process itself differs from Poissonian. Experiments are currently being carried out to test the model over a wider range of conditions, and will be reported subsequently.

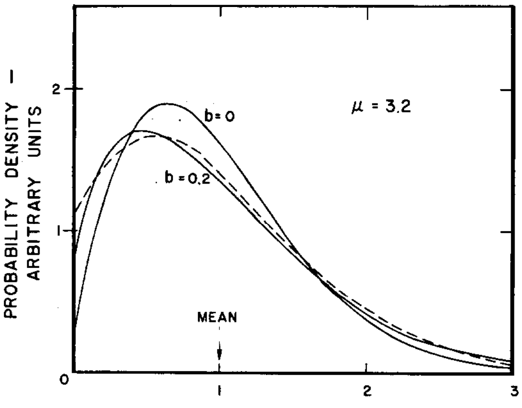

Fig. 9. Comparison of an experimental single-electron spectrum for an EMI 9514A photomultiplier (dashed line) with computed single-electron spectra with $b = 0$ and 0.2 (solid lines).

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